DUALITY IN TENSOR-TRIANGULAR GEOMETRY
LECTURE NOTES

Overview. Tensor-triangular geometry provides a framework for studying classification problems across a range of areas. Originating in chromatic homotopy theory, its influence has since been seen in commutative algebra, geometry, and representation theory. In the course, we will see how one can prove powerful duality statements from a tensor-triangular perspective. We will focus on examples in commutative algebra, but the theory developed is rather broad, so other examples may be covered depending on the audience's interests.

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Relevant literature: main references.

(1) P. Balmer, I. Dell'Ambrogio, and B. Sanders. Grothendieck-Neeman duality and the Wirthmüller isomorphism. *Compos. Math.*, 152(8):1740–1776, 2016
(2) H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:No. 4, 107–131, 2003
(3) T. Peirce, J. Williamson. Duality in tensor-triangular geometry via proxy-smallness. arXiv:2510.24415, 2025

Related literature: other helpful references.

(1) H. Krause. Localization theory for triangulated categories. In Triangulated categories, volume 375 of London Math. Soc. Lecture Note Ser., pages 161–235. Cambridge Univ. Press, Cambridge, 2010
(2) H. Krause. Homological theory of representations, volume 195 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2022
(3) M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. Mem. Amer. Math. Soc., 128(610):x+114, 1997
(4) C. Weibel. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics. 38. Cambridge: Cambridge University Press. xiv, 450 p. (1994).
(5) A. Neeman. Triangulated categories. Annals of Mathematics Studies. 148. Princeton, NJ: Princeton University Press. vii, 449 p. (2001).

Assessment. The final exam will be an oral exam. For zápočet, students are expected to participate during the course.

1. Recollections on triangulated categories

1.A. **Axioms.** Loosely speaking, a triangulated category consists of an additive category T, together with two extra pieces of data:

(1) an equivalence of categories $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$ called the *shift*;
(2) a collection of *triangles* $X \to Y \to Z \to \Sigma X$ satisfying various axioms which ensure good behaviour.

Definition 1.1. Let T be an additive category and $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$ be an additive equivalence of categories. A *candidate triangle* is a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

such that the composites $g \circ f$, $h \circ g$, and $\Sigma f \circ h$ are all zero. A morphism of candidate triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'. \end{tikzcd}$$

Considering a smaller class of candidate triangles which satisfy certain properties leads to the notion of a triangulated category. The first four axioms are easy to justify, but the final axiom is harder to motivate. It is convenient to develop the theory assuming only these first four axioms, and then add in the final one once it becomes relevant. Nonetheless, we'll give both definitions now, so that we can consider an example before embarking on the abstract theory.

Definition 1.2. A *pretriangulated category* T is an additive category together with an additive equivalence of categories $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$, and a subclass of candidate triangles called *distinguished triangles* which satisfy the following axioms:

(TR0) Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle, and for all $X \in \mathsf{T}$ the candidate triangle

$$X \xrightarrow{1} X \to 0 \to \Sigma X$$

is distinguished.

(TR1) For all $f\colon X \to Y$ in T, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$.

(TR2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a candidate triangle. This is distinguished if and only if the candidate triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

(TR3) For any commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] & \Sigma X \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'. \end{tikzcd}$$

in which the rows are triangles, there exists a map $w\colon Z \to Z'$ (which need *not* be unique) making

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'.
\end{tikzcd}$$

commute.

Remark 1.3. It is standard to drop the adjective ‘distinguished’, and just refer to them as *triangles*. For candidate triangles, we will never drop the adjective.

Remark 1.4. By combining (TR2) with (TR3), one sees that there always exists fillers in the first and second column too.

Definition 1.5. A *triangulated category* T is a pretriangulated category satisfying the following additional axiom:

(TR4) Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z' \xrightarrow{h} \Sigma X$, $Y \xrightarrow{u} Z \xrightarrow{v} X' \xrightarrow{w} \Sigma Y$ and $X \xrightarrow{uf} Z \xrightarrow{a} Y' \xrightarrow{b} \Sigma X$ are distinguished triangles. Then there exists a distinguished triangle

$$Z' \xrightarrow{\alpha} Y' \xrightarrow{\beta} X' \xrightarrow{\gamma} \Sigma Z'$$

such that $v = \beta a$, $h = b\alpha$, $\gamma = \Sigma g \circ w$, $w\beta = \Sigma f \circ b$ and $\alpha g = au$.

Pictorially this axiom can be represented by the following commuting diagram:

$$\begin{tikzcd}
 & & Y' \arrow[rrd, red, "\beta"] \arrow[ddll, dashed, "b"] & & \\
Z' \arrow[urr, red, "\alpha"] \arrow[d, dashed, "h"'] & & & & X' \arrow[llll, dashed, "\Sigma g\circ w"] \\
X \arrow[rrrr, "uf"] \arrow[drr, "f"'] & & & & Z \arrow[u, "v"'] \arrow[uull, "a"'] \arrow[dll, dashed, "w"'] \\
 & & Y \arrow[uull, "g"] \arrow[urr, "u"'] & &
\end{tikzcd}$$

The dotted maps are of degree 1 (i.e., $f\colon X \dashrightarrow Y$ represents a map $f\colon X \to \Sigma Y$), and composites of the form $\to\to\dashrightarrow$ are triangles. The red maps are the extra data, together with the condition that they form a triangle. In order to remember this, note that the primed letters are the cones of maps, and that every triangle contains an X, Y, and a Z (primed or otherwise). In light of the shape of the above diagram, (TR4) is often referred to as the *octahedral axiom*.

Alternatively, one can give the following pictorial representation.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[r, "g"] \arrow[d, "u"] & Z' \arrow[r, "h"] \arrow[d, dashed, "\alpha"] & \Sigma X \arrow[d, "1"] \\
X \arrow[r, "uf"] \arrow[d, "f"] & Z \arrow[r, "a"] \arrow[d, "1"] & Y' \arrow[r, "b"] \arrow[d, dashed, "\beta"] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, "u"] \arrow[d, "g"] & Z \arrow[r, "v"] \arrow[d, "a"] & X' \arrow[r, "w"] \arrow[d, "1"] & \Sigma Y \arrow[d, "\Sigma g"] \\
Z' \arrow[r, dashed, "\alpha"] & Y' \arrow[r, dashed, "\beta"] & X' \arrow[r, dashed, "\gamma"] & \Sigma Z'
\end{tikzcd}$$

The first three rows are the given triangles, and (TR4) then asserts the existence of the dotted arrows making the diagram commute, so that the bottom row is also a triangle.

Let's briefly discuss the axioms and provide some motivation for them. If one thinks as triangles as a generalisation of short exact sequences, then in a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ you should think of Z as the (homotopy coherent) cokernel of f, and X as the (homotopy coherent) kernel of g. The axioms then mean the following.

- (TR0) The kernel and cokernel of the identity is zero.
- (TR1) Every map has a kernel and cokernel.
- (TR2) Up to sign, every map is the kernel of its cokernel and vice versa.
- (TR3) Kernels and cokernels are almost functorial.
- (TR4) One can interpret the given triangles as saying $Z' \simeq Y/X$, $X' \simeq Z/Y$ and $Y' \simeq Z/X$, and then the axiom asserts that $X' \simeq Y'/Z'$, i.e., $(Z/X)/(Y/X) \simeq Z/Y$.

Remark 1.6. Given a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$, it is common to call Z the *cofibre* (or *cone*) of f, and X the *fibre* (or *cocone*) of g. Sometimes it is customary to write triangles as $X \to Y \to Z$ and drop the map to the shift. We will sometimes subscribe to this later on the course for brevity, but we warn the reader that it is important not to forget this map. For example, a morphism of triangles requires the square including the map $Z \to \Sigma X$ to commute.

Let us recall some key features:

(1) Products and coproducts of triangles are again triangles (provided the required (co)products exist).
(2) If $X \to Y \to Z \to \Sigma X$ is a triangle, then $Z \simeq 0$ if and only if $X \to Y$ is an isomorphism.
(3) Given a commutative diagram of pretriangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

if f and g are isomorphisms then h is also an isomorphism. Note that this gives a good way to show that a pretriangle is infact a triangle, since triangles are closed under isomorphism.

1.B. **Examples.** We briefly describe three examples of triangulated categories.

1.B.1. *The homotopy category of a ring.* Fix a ring R. Recall that a map $f \colon M \to N$ of chain complexes of (left) R-modules is null homotopic if it is chain homotopic to the zero map. We write $\mathrm{Null}(M, N)$ for the subgroup of $\mathrm{Hom}_{\mathrm{Ch}(R)}(M, N)$ consisting of the null homotopic maps.

The *homotopy category* $\mathsf{K}(R)$ is defined by having objects the chain complexes of R-modules, and morphisms given by the homotopy classes of chain maps, i.e.,

$$\mathrm{Hom}_{\mathsf{K}(R)}(M,N) = \mathrm{Hom}_{\mathrm{Ch}(R)}(M,N)/\mathrm{Null}(M,N).$$

Equivalently, the morphisms are the chain maps up to the equivalence relation of chain homotopy equivalence. The distinguished triangles in the homotopy category are the triangles which are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopic) to those of the form

$$M \xrightarrow{f} N \to C(f) \to \Sigma M$$

for some map of chain complexes $f\colon M \to N$.

1.B.2. *The derived category of a ring.* Homological algebra is built upon taking resolutions of modules. Therefore, one seeks a category which contains precisely the homological information of modules, so that objects are resolutions, and a module is isomorphic to any resolution of it. As such, we want a category which contains all the resolutions of modules, and in which quasi-isomorphisms are isomorphisms. Since injective resolutions and projective resolutions point in opposite directions, we consider all chain complexes (i.e., rather than just those bounded above or below 0). The *derived category* of a ring R is the universal category in which quasi-isomorphisms of complexes are inverted. We denote this category by $\mathsf{D}(R) := \mathsf{K}(R)[\text{quasi isos}^{-1}]$. We now give a precise construction of this.

Remark 1.7. This construction of the derived category leads to some set-theoretic discussions, namely, why are the hom sets actually sets? For the purposes of this course we ignore this, and just remark that one can give alternative constructions bypassing this issue.

The objects of $\mathsf{D}(R)$ are the same as the objects of $\mathsf{K}(R)$, that is, they are the chain complexes of R-modules. The morphisms in $\mathsf{D}(R)$ are equivalence classes of rooves, defined as follows. Let $M, N \in \mathsf{D}(R)$. A *roof* from M to N is a pair of chain maps

$$\begin{tikzcd} & Z \arrow[dl, "s"', "\simeq"] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where $Z \in \mathsf{D}(R)$ and s is a quasi-isomorphism. Two rooves $(M \leftarrow Z \to N)$ and $(M \leftarrow Z' \to N)$ are equivalent if there exists another roof $(M \leftarrow W \to N)$ and maps $W \to Z$ and $W \to Z'$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ M & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & N \\ & Z' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. The hom sets of $\mathsf{D}(R)$ are rooves up to this equivalence relation.

Remark 1.8. Defining composites requires a little work: given two rooves $(L \xleftarrow{s} Z \xrightarrow{f} M)$ and $(M \xleftarrow{s'} Z' \xrightarrow{g} N)$, their composite gf is defined as follows. We have a triangle

$$Z' \xrightarrow{s'} M \xrightarrow{\alpha} C$$

and another triangle

$$F \xrightarrow{\beta} Z \xrightarrow{\alpha f} C.$$

By the existence of fillers we have a diagram

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[d, dashed, "\gamma'"] & Z \arrow[r, "\alpha f"] \arrow[d, "f"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & \Sigma F \arrow[d, dashed, "\Sigma\beta"] \\ Z' \arrow[r, "s'"'] & M \arrow[r, "\alpha"] & C \arrow[r] & \Sigma Z' \end{tikzcd}$$

Since s' is a quasi-isomorphism, C is acyclic so β is also a quasi-isomorphism. The composite is then given by the roof

$$L \xleftarrow{s\beta} F \xrightarrow{g\gamma} N.$$

A tedious check shows everything is well-defined up to equivalence of rooves.

We end by recalling some key features of the derived category:

(1) If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is short exact sequence of R-modules, then there exists a chain map $N \to \Sigma L$ such that $L \xrightarrow{f} M \xrightarrow{g} N \to \Sigma L$ is a triangle in $\mathsf{D}(R)$.
(2) In fact, one can generalise the previous to see that short exact sequences of complexes also give triangles in the derived category.
(3) Let M and N be R-modules. Then

$$\mathrm{Hom}_{\mathsf{D}(R)}(M, \Sigma^i N) = \mathrm{Ext}^i_R(M, N).$$

(4) For any $M \in \mathsf{D}(R)$ we have $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, M) = H_i M$.

1.B.3. *The stable module category.* Let G be a finite group and let k be a field of characteristic dividing the order of G. In this setting, Maschke's theorem fails and we are in the realm of *modular* representation theory. A key category of interest in this area is the *stable module category* $\mathsf{StMod}(kG)$.

Lemma 1.9. *A kG-module M is projective if and only if it is injective.*

Proof. We first show that kG is self-injective. Define a k-linear map $\pi\colon kG \to k$ by $\pi(g) = 1$ if $g = 1$ and $\pi(g) = 0$ otherwise. One can check that the map $kG \to \mathrm{Hom}_k(kG, k)$ defined by $g \mapsto [h \mapsto \pi(g^{-1}h)]$ is an isomorphism of kG-modules. Now

$$\mathrm{Hom}_{kG}(-, kG) = \mathrm{Hom}_{kG}(-, \mathrm{Hom}_k(kG, k)) = \mathrm{Hom}_k(\mathrm{res}^{kG}_k(-), k)$$

which is exact, hence kG is an injective kG-module.

Since kG is Noetherian, direct sums of injective modules are injective. Therefore free modules are injectives, and hence summands of free modules (i.e., projectives) are also injective. Conversely, suppose that M is injective. The unit of the induction-restriction adjunction $M \to kG \otimes_k \mathrm{res}^{kG}_k M$ is injective and hence splits. Now $kG \otimes_k \mathrm{res}^{kG}_k M$ is a free module, so M is projective. $\square$

The objects of $\mathsf{StMod}(kG)$ are the kG-modules, and the morphisms are given by the kG-module maps modulo those which factor through some projective:

$$\mathrm{Hom}_{\mathsf{StMod}(kG)}(M, N) = \frac{\mathrm{Hom}_{kG}(M, N)}{P\mathrm{Hom}_{kG}(M, N)}$$

where $P\mathrm{Hom}_{kG}(M, N)$ denotes the linear subspace of homomorphisms which factor through some projective kG-module.

The shift functor is given by taking cosyzygies: given X we define ΣX by taking an exact sequence

$$0 \to X \to E \to \Sigma X \to 0$$

where E is injective. To show that this is an autoequivalence, one constructs Σ^{-1} by taking syzygies; here is where Lemma 1.9 comes in. Any short exact sequence $0 \to X \to Y \to Z \to 0$ fits into a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[r, "\beta"] \arrow[d] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X \arrow[r] & E \arrow[r] & \Sigma X \arrow[r] & 0
\end{tikzcd}$$

where E is injective. The triangles are those isomorphic to ones of the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

as in the above diagram.

We end by recalling some key features of the stable module category:

(1) We have $M \cong N$ in $\mathsf{StMod}(kG)$ if and only if there exists projectives P and Q such that $M \oplus P \cong N \oplus Q$ in $\mathsf{Mod}(kG)$.
(2) The homomorphisms in the stable module category compute Ext-groups: if $i \geq 0$ then

$$\mathrm{Hom}_{\mathsf{StMod}(kG)}(M, \Sigma^i N) = \mathrm{Ext}^i_{kG}(M, N).$$

In particular, taking $M = k$ with trivial G-action, it computes group cohomology: $H^i(G; N) = \mathrm{Hom}_{\mathsf{StMod}(kG)}(k, \Sigma^i N)$. (In fact, allowing negative i too, one obtains Tate cohomology groups.) Note that here unlike in the derived category, we are representing the elements of Ext by actual module maps, rather than by using complexes.

1.B.4. *Some other examples.*

- The derived category $\mathsf{D}_{\mathrm{qx}}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology for a nice enough scheme X.
- The (equivariant) stable homotopy category of spectra (representing objects for (equivariant) cohomology theories for topological spaces).
- The Kasparov category of separable G-C^*-algebras.

2. Tensor-triangulated categories

Recall that a symmetric monoidal structure on a category $\mathcal{C}$ is the data of a bifunctor

$$- \otimes - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

called the *tensor product* (or *monoidal product*) together with an unit object $\mathbb{1} \in \mathcal{C}$ satisfying:

- unitality: $\mathbb{1} \otimes X \simeq X$ for all $X \in \mathcal{C}$;
- associativity: $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- symmetry: $X \otimes Y \simeq Y \otimes X$ for all $X, Y \in \mathcal{C}$;

all satisfying various coherences that we won't make precise here. Such a monoidal structure is moreover called *closed*, if there is a bifunctor $\underline{\mathrm{Hom}}(-,-) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ called the *internal hom* such that $- \otimes X$ is left adjoint to $\underline{\mathrm{Hom}}(X, -)$. Again, this is all subject to various coherences that we will not make explicit. Note that $\underline{\mathrm{Hom}}(\mathbb{1}, X) \simeq X$ for all $X \in \mathcal{C}$.

In order to blend this data with that of a triangulated category we need one more definition:

Definition 2.1. Let T and U be triangulated categories. A *triangulated functor* is an additive functor $F \colon \mathsf{T} \to \mathsf{U}$ together with a natural isomorphism $\varphi \colon F\Sigma \xrightarrow{\sim} \Sigma F$ such that for any triangle

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in T, the candidate triangle

$$FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \xrightarrow{\varphi_X \circ F(h)} \Sigma FX$$

is a triangle in U.

With this, we can make the key definition:

Definition 2.2. A *tensor-triangulated category* T (tt-category for short) is a triangulated category which also has a closed symmetric monoidal structure which is compatible in the following sense: the tensor product is a triangulated functor in both variables, and the internal hom is a triangulated functor in both variables.

Remark 2.3. In fact, we only require that $\underline{\mathrm{Hom}}(-,X)$ is triangulated up to sign. We ignore this distinction, since for most purposes the signs are irrelevant.

Definition 2.4. A functor $F\colon \mathsf{T} \to \mathsf{U}$ between tt-categories is a *tt-functor* if it is triangulated and strong monoidal. (Recall that strong monoidality means that there are natural isomorphisms $F(X \otimes Y) \simeq FX \otimes FY$ and $F(\mathbb{1}_\mathsf{T}) \simeq \mathbb{1}_\mathsf{U}$ satisfying various coherences.)

2.A. **Examples.** We record two key examples.

2.A.1. *The derived category of a commutative ring.* If R is a commutative ring then $\mathsf{D}(R)$ is a tensor-triangulated category. The tensor product is given by the derived tensor product $- \otimes^{\mathsf{L}}_R -$ and the internal hom is given by the derived hom $\mathsf{R}\mathrm{Hom}_R(-,-)$. These are defined as follows. Note that for an arbitrary $M \in \mathsf{D}(R)$, the functors $M \otimes_R -$ and $\mathrm{Hom}_R(M,-)$ do not preserve quasi-isomorphisms, so cannot induce a functor on the derived category. This is the case even if M is a complex of projective R-modules:

Example 2.5. Consider $R = \mathbb{Z}/4$ and the complex

$$M = \cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \cdots .$$

The quasi-isomorphism $M \xrightarrow{\sim} 0$ is not preserved by $\mathrm{Hom}_{\mathbb{Z}/4}(M,-)$.

Instead, we use *dg-projective resolutions*: a dg-projective resolution for $M \in \mathsf{D}(R)$ consists of a quasi-isomorphism $P \xrightarrow{\sim} M$ where P has the property that $\mathrm{Hom}_R(P,-)$ preserves surjective quasi-isomorphisms. The derived tensor product is then defined by $M \otimes^{\mathsf{L}}_R - = P \otimes_R -$ where P is a dg-projective resolution of M. Likewise $\mathsf{R}\mathrm{Hom}_R(M,-) = \mathrm{Hom}_R(P,-)$. There is an analogous notion of dg-injective resolution used to define $\mathsf{R}\mathrm{Hom}_R(-,M)$.

2.A.2. *The stable module category.* Let G be a finite group and k be a field of characteristic dividing the order of G. Then the stable module category $\mathsf{StMod}(kG)$ is a tensor-triangulated category. The tensor product is given by $- \otimes_k -$ with the diagonal G-action, with monoidal unit k with the trivial action. To see that this is indeed well-defined on the stable category, let us first consider what happens on $\mathsf{Mod}(kG)$. Firstly note that $M \otimes_k N \simeq N \otimes_k M$; one easily checks that this is indeed kG-linear. One can make $\mathrm{Hom}_k(M,N)$ into a kG-module via $(g \cdot f)(m) = gf(g^{-1}m)$. In particular, one sees that $g \cdot f = f$ for all $g \in G$ if and only if $f(gm) = gf(m)$ for all g, m. Therefore,

$$\mathrm{Hom}_k(M,N)^G = \mathrm{Hom}_{kG}(M,N).$$

From this, one checks that one has the tensor-hom adjunction

$$\mathrm{Hom}_{kG}(L \otimes_k M, N) = \mathrm{Hom}_{kG}(L, \mathrm{Hom}_k(M,N)).$$

From this the following is easy:

Lemma 2.6. *For any kG-module M, if P is a projective kG-module, then $P \otimes_k M$ is again projective.*

Proof. We have

$$\mathrm{Hom}_{kG}(P \otimes_k M, -) \simeq \mathrm{Hom}_{kG}(P, \mathrm{Hom}_k(M, -))$$

by the tensor-hom adjunction. The latter is an exact functor, hence $P \otimes_k M$ is projective. $\square$

Since the tensor product on Mod_{kG} preserves projectives, it descends to a tensor product on $\mathsf{StMod}(kG)$. In a similar way one sees that $\mathrm{Hom}_k(M, -)$ preserves injectives, and hence also passes to give a functor on $\mathsf{StMod}(kG)$. Therefore, $\mathsf{StMod}(kG)$ is indeed a tt-category as claimed.

2.B. **Smallness in tt-categories.** Let us first briefly return to the setting of a triangulated category T (i.e., not necessarily with a tensor product). We study certain types of 'small' objects in triangulated categories. It is often the case that the whole triangulated category is generated by small objects, and much of the theory of triangulated categories relies upon such assumptions.

Definition 2.7. Let T be a triangulated category which has coproducts. An object $X \in \mathsf{T}$ is said to be *compact* if the natural map

$$\bigoplus \mathrm{Hom}_{\mathsf{T}}(X, Y_i) \to \mathrm{Hom}_{\mathsf{T}}(X, \bigoplus Y_i)$$

is an equivalence for every set of objects $\{Y_i\}$. We write T^c for the full subcategory of T consisting of the compact objects.

Before we can give some examples of compact objects, and criteria for detecting them, we must introduce some terminology.

Definition 2.8. Let T be a triangulated category which has coproducts.

(1) A full subcategory $\mathcal{S}$ of T is *thick* if it is closed under retracts and is triangulated.
(2) A full subcategory $\mathcal{S}$ of T is *localizing* if it is thick, and closed under coproducts.

Given a set of objects $\mathcal{X}$ of T, we write $\mathrm{Thick}(\mathcal{X})$ (resp., $\mathrm{Loc}(\mathcal{X})$) for the smallest thick (resp., localizing) subcategory of T containing $\mathcal{X}$. Note that these are well defined since the intersection of thick/localizing subcategories is again thick/localizing.

Example 2.9. Let $X \in \mathsf{T}$. The full subcategory $\{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i X, Y) \simeq 0 \text{ for all } i \in \mathbb{Z}\}$ is thick. It is localizing if X is compact.

Definition 2.10. Let $\mathcal{X}$ be a set of objects of T. We say that $\mathcal{X}$ *generates* T if $\mathrm{Loc}(\mathcal{X}) = \mathsf{T}$. If each element of $\mathcal{X}$ is compact, then we say that $\mathcal{X}$ *compactly generates* T.

Let us give an alternative characterisation of compact generation:

Proposition 2.11. *Let T be a triangulated category with coproducts, and $\mathcal{S}$ be a set of compact objects of T. Then the following are equivalent:*

(1) $\mathcal{S}$ generates T;
(2) if $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, X) = 0$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$, then $X \simeq 0$.

Proof. Left as Exercise A.1. $\square$

If T is compactly generated, then we can characterise the compact objects in terms of building operations.

Proposition 2.12. *Let* T *be a triangulated category which is compactly generated by a set* G*. Then* $\mathsf{T}^{\mathrm{c}} = \mathrm{Thick}(\mathsf{G})$*.*

Proof. The implication that $X \in \mathsf{T}^{\mathrm{c}}$ implies that $X \in \mathrm{Thick}(\mathsf{G})$ requires some work so we omit it. For the converse, consider the set

$$\{Y \in \mathsf{T} \mid \bigoplus \mathrm{Hom}_{\mathsf{T}}(Y, Z_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(Y, \bigoplus Z_i) \text{ for all sets } \{Z_i\}\}.$$

This is a thick subcategory of T, and contains G. Hence it contains $\mathrm{Thick}(\mathsf{G})$ since this is the *smallest* thick subcategory of T containing G. $\square$

Example 2.13. We have $\mathsf{D}(R)^{\mathrm{c}} = \mathsf{Perf}(R)$ where the right hand side denotes the full subcategory of *perfect complexes*, that is, those complexes which are quasi-isomorphic to a bounded complex of finitely generated projectives. We leave the proof of this as Exercise A.2.

Example 2.14. We have $\mathsf{StMod}(kG)^{\mathrm{c}} = \mathsf{stmod}(kG)$ where the right hand side denotes the full subcategory of finitely generated modules. If M is compact, then writing M as a direct limit $M = \varinjlim M_i$ of finitely generated modules, we have $\mathrm{Hom}(M, M) = \varinjlim \mathrm{Hom}(M, M_i)$. In particular, the identity on M factors through M_i so that M_i is a retract of a finitely generated and hence is finitely generated. The other inclusion is left as Exercise A.3.

If T is a tt-category, one might hope that T^{c} inherits these properties. Unfortunately, this need not always be the case: the tensor product of compacts need not be compact in general, and the unit need not be compact in general.

Example 2.15. Let $\mathsf{T} = \prod_i \mathsf{T}_i$ be the countable product of tt-categories T_i with tt-structure given pointwise. Then a tuple $(X_i) \in \mathsf{T}$ is compact if and only if for almost all i we have $X_i = 0$ and $X_j \in \mathsf{T}_j^c$ for all j. In particular, since $\mathbb{1}_{\mathsf{T}} = (\mathbb{1}_{\mathsf{T}_i})$ the unit is not compact.

There is an alternative subcategory of 'small' object which behaves better with the tensor structure: Recall that the counit of the tensor-hom adjunction is the evaluation map $\mathrm{ev}\colon X \otimes \underline{\mathrm{Hom}}(X, Y) \to Y$ and the unit is the coevaluation map $\mathrm{coev}\colon Y \to \underline{\mathrm{Hom}}(X, X \otimes Y)$.

Definition 2.16. Suppose that T is a tensor-triangulated category with coproducts.

(1) An object $X \in \mathsf{T}$ is said to be *rigid* if the natural map
$$\nu_{X,Y}\colon \underline{\mathrm{Hom}}(X, \mathbb{1}) \otimes Y \to \underline{\mathrm{Hom}}(X, Y)$$
is an equivalence for all $Y \in \mathsf{T}$. To spell it out, the natural map is the composite
$$\underline{\mathrm{Hom}}(X, \mathbb{1}) \otimes Y \xrightarrow{\mathrm{coev}} \underline{\mathrm{Hom}}(X, X \otimes \underline{\mathrm{Hom}}(X, \mathbb{1}) \otimes Y) \xrightarrow{\underline{\mathrm{Hom}}(X, \mathrm{ev})} \underline{\mathrm{Hom}}(X, Y).$$

(2) An object $X \in \mathsf{T}$ is said to be *internally compact* if the natural map
$$\bigoplus \underline{\mathrm{Hom}}(X, Y_i) \to \underline{\mathrm{Hom}}(X, \bigoplus Y_i)$$
is an equivalence for every set $\{Y_i\}$.

Example 2.17. For any tt-category T, the tensor unit $\mathbb{1}$ is both rigid and internally compact.

In order to study the properties of rigid objects it is helpful to consider an alternative definition (which we will show is in fact equivalent). We write $D\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ for the functor $D = \underline{\mathrm{Hom}}(-, \mathbb{1})$. This is called the *functional dual* (or sometimes Spanier-Whitehead dual).

Definition 2.18. Let T be a tensor-triangulated category, and $X \in \mathsf{T}$. The object X is *dualizable* if and only if there exists a map $\eta\colon \mathbb{1} \to X \otimes DX$ such that the diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[dr, "\mathrm{coev}"'] & X \otimes DX \arrow[d, "\nu_{X,X}"] \\ & \underline{\mathrm{Hom}}(X,X) \end{tikzcd}$$

commutes.

Firstly, note that there is a natural map $\rho\colon X \to D^2X$ given by the composite

$$X \xrightarrow{\mathrm{coev}} \underline{\mathrm{Hom}}(DX, X \otimes DX) \xrightarrow{\underline{\mathrm{Hom}}(DX,\mathrm{ev})} \underline{\mathrm{Hom}}(DX, \mathbb{1})$$

for all $X \in \mathsf{T}$. Many of the proofs of the following statements follow by tedious diagram chases, so we provide only an outline of the proofs.

Lemma 2.19. *Let T be a tensor-triangulated category and $X \in \mathsf{T}$. If X is dualizable then DX is dualizable.*

Proof. We define a map $\mathbb{1} \to DX \otimes D^2X$ via the composite

$$\mathbb{1} \xrightarrow{\eta} X \otimes DX \xrightarrow{\rho \otimes DX} D^2X \otimes DX.$$

It remains to check that the required diagram commutes. That is, we want to check that the outer square in the diagram

$$\begin{tikzcd} \mathbb{1} \arrow[rr, "\eta"] \arrow[dr, "\mathrm{coev}"] \arrow[dd, "\mathrm{coev}"'] & & X \otimes DX \arrow[dl, "\nu_{X,X}"'] \arrow[dd, "\rho\otimes DX"] \\ & \underline{\mathrm{Hom}}(X,X) \arrow[dl, "\alpha"] & \\ \underline{\mathrm{Hom}}(DX,DX) & & D^2X \otimes DX \arrow[ll, "\nu_{DX,DX}"] \end{tikzcd}$$

commutes, where α is defined to be the composite

$$\underline{\mathrm{Hom}}(X,X) \xrightarrow{\underline{\mathrm{Hom}}(X,\rho)} \underline{\mathrm{Hom}}(X, D^2X) \simeq \underline{\mathrm{Hom}}(X \otimes DX, \mathbb{1}) \simeq \underline{\mathrm{Hom}}(DX, DX).$$

The top triangle commutes since X is dualizable, so it suffices to prove that the other subdiagrams commute. This can be checked by unravelling the definitions of α and ρ; we leave this as a (tedious) exercise for the reader. □

Lemma 2.20. *Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is dualizable. Then the natural map $\rho\colon X \to D^2X$ is an equivalence.*

Proof. We claim that the composite

$$D^2X \simeq \mathbb{1} \otimes D^2X \xrightarrow{\eta \otimes D^2X} X \otimes DX \otimes D^2X \xrightarrow{X \otimes \mathrm{ev}} X$$

is an inverse to ρ. This can be checked by diagram chasing. □

Proposition 2.21. *Let T be a tensor-triangulated category, and $X \in \mathsf{T}$. Then X is rigid if and only if it is dualizable.*

Proof. The forward direction is clear since we may take $\eta = \nu_{X,X}^{-1} \circ \mathrm{coev}$. For the reverse direction, we claim that the composite

$$\underline{\mathrm{Hom}}(X,Y) \simeq \underline{\mathrm{Hom}}(X,Y) \otimes \mathbb{1} \xrightarrow{\underline{\mathrm{Hom}}(X,Y)\otimes\eta} \underline{\mathrm{Hom}}(X,Y) \otimes X \otimes DX \xrightarrow{\mathrm{ev}\otimes DX} Y \otimes DX$$

is an inverse to $\nu_{X,Y}$. Again, one may verify this by diagram chasing. $\square$

Remark 2.22. By Lemma 2.20, one deduces that if X is a rigid object, then we have adjunctions $X \otimes - \dashv DX \otimes - \dashv X \otimes -$.

Lemma 2.23. *Let $F\colon \mathsf{T} \to \mathsf{U}$ be a tt-functor between tt-categories. Then F preserves rigid (=dualizable) objects. Moreover, if X is rigid, then $F(DX) \simeq D(FX)$.*

Proof. For any $X \in \mathsf{T}$, we can construct a map $\varphi\colon F(DX) \to D(FX)$ as the adjunct to

$$FX \otimes F(DX) \simeq F(X \otimes DX) \xrightarrow{F(\mathrm{ev})} F(\mathbb{1}_{\mathsf{T}}) \simeq \mathbb{1}_{\mathsf{U}}.$$

If X is dualisable, then we have a map $\eta\colon \mathbb{1}_{\mathsf{T}} \to X \otimes DX$ and hence we have a map $\eta'\colon \mathbb{1}_{\mathsf{U}} \to FX \otimes D(FX)$ defined as the composite

$$\mathbb{1}_{\mathsf{U}} \simeq F(\mathbb{1}_{\mathsf{T}}) \xrightarrow{F(\eta)} F(X \otimes DX) \simeq FX \otimes F(DX) \xrightarrow{FX \otimes \varphi} FX \otimes D(FX).$$

Using this one can verify that FX is dualisable.

To see that the map $\varphi\colon F(DX) \to D(FX)$ constructed above is an isomorphism, one can construct an inverse as follows. Since FX is dualisable, $D(FX)$ is also dualisable (Lemma 2.19) so by adjunction maps $D(FX) \to F(DX)$ correspond to maps $\mathbb{1}_{\mathsf{U}} \to FX \otimes F(DX)$. We can construct such a map as the composite

$$\mathbb{1}_{\mathsf{U}} \simeq F(\mathbb{1}_{\mathsf{T}}) \xrightarrow{F(\eta)} F(X \otimes DX) \simeq FX \otimes F(DX).$$

One can check directly that this gives an inverse to φ. $\square$

Is it natural to ask how the different notions of 'smallness' in Definition 2.7 and Definition 2.16 are related. The following two results answer this. We also note the following relation, which does not rely on a set of rigid generators.

Proposition 2.24. *Let T be a tensor-triangulated category. If $\mathbb{1}$ is compact, then any rigid object is compact.*

Proof. Let X be rigid. Then $\mathrm{Hom}(X, -) \simeq \mathrm{Hom}(\mathbb{1}, DX \otimes -)$. Since $\mathbb{1}$ is compact this commutes with coproducts showing that X is compact. $\square$

Proposition 2.25. *Let T be a tensor-triangulated category, and suppose that T has a set of rigid generators G.*

(1) For $X \in \mathsf{T}$ we have

$$X \in \mathrm{Thick}(\mathsf{G}) \implies X \text{ is rigid} \iff X \text{ is internally compact.}$$

(2) If the elements of G are compact, then for $X \in \mathsf{T}$ we have

$$X \text{ is compact} \iff X \in \mathrm{Thick}(\mathsf{G}) \implies X \text{ is rigid} \iff X \text{ is internally compact.}$$

(3) If the elements of G are compact, and the unit $\mathbb{1}$ of T is compact, then for $X \in \mathsf{T}$ we have

$$X \text{ is compact} \iff X \in \mathrm{Thick}(\mathsf{G}) \iff X \text{ is rigid} \iff X \text{ is internally compact.}$$

Proof. For (1), firstly note that the set of rigid objects in T is thick. By assumption it contains G, so we have $\mathrm{Thick}(\mathsf{G}) \subseteq \{\text{rigid objects}\}$, which proves the first implication. That rigid objects are internally compact is an immediate consequence of the definitions. For the remaining implication, suppose X is internally compact, and consider the set

$$\mathcal{L} = \{Y \in \mathsf{T} \mid DX \otimes Y \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, Y)\}.$$

The set $\mathcal{L}$ is localizing as X is internally compact, and contains G by Exercise A.4. Therefore $\mathcal{L} = \mathsf{T}$, and hence X is rigid.

For (2), it suffices to prove that X is compact if and only if $X \in \mathrm{Thick}(\mathsf{G})$, which was the content of Proposition 2.12. (3) follows from Proposition 2.24. ☐

Definition 2.26. Let T be a compactly generated tt-category. We say that T is *rigidly-compactly generated* if the compact and rigid objects in T coincide. Note that by Proposition 2.25(3) this is equivalent to T having a set of rigid and compact generators and the tensor unit being compact.

Example 2.27. Let T be a tt-category which is compactly generated by its tensor unit. Then T is rigidly-compactly generated. For example, $\mathsf{D}(R)$ is rigidly-compactly generated.

Example 2.28. The stable module category $\mathsf{StMod}(kG)$ is rigidly-compactly generated. Since the compacts are the finitely generated modules it is clear that it is compactly generated. It is easy to see that the compacts coincide with the rigid objects (since if M is finitely generated over kG then it has finite rank over k).

Example 2.29. Let us revisit Example 2.15: we consider $\mathsf{T} = \prod_i \mathsf{T}_i$ where each T_i is rigidly-compactly generated by G_i. The set $\mathsf{G} = \{\delta_i(G_i) \mid G_i \in \mathsf{G}_i\}$ of "characteristic objects" given by $(\delta_i X)_i = X_i$ and $(\delta_i X)_j = 0$ for $i \neq j$ is a set of rigid generators for T. Moreover these are compact. So we are in case (2) of Proposition 2.25. However, we already saw that the unit is not compact (but is rigid). So the inclusion $\mathsf{T}^c \subset \{\text{rigids}\}$ is proper.

Finally, it is convenient to introduce terminology for the appropriate analogues of thick and localizing subcategories in the tensor-triangular setting.

Definition 2.30. Let T be a tensor-triangulated category. A thick/localizing subcategory $\mathcal{S}$ is a *$\otimes$-ideal* if for all $X \in \mathcal{S}$ and $Y \in \mathsf{T}$ we have $X \otimes Y \in \mathcal{S}$. We write $\mathrm{Thick}^{\otimes}(-)$ and $\mathrm{Loc}^{\otimes}(-)$ for the smallest thick and localizing $\otimes$-ideal generated by a set of objects.

Proposition 2.31. *Let T be a tensor-triangulated category and $X \in \mathsf{T}$ be rigid.*

(1) If Y is compact, then $X \otimes Y$ is compact.
(2) If Y is internally compact, then $X \otimes Y$ is internally compact.
(3) If Y is rigid, then $X \otimes Y$ is rigid.

Proof. For (1), we isomorphisms

$$\mathrm{Hom}(X \otimes Y, \bigoplus Z_i) = \mathrm{Hom}(X, DY \otimes \bigoplus Z_i) = \bigoplus \mathrm{Hom}(X, DY \otimes Z_i) = \bigoplus \mathrm{Hom}(X \otimes Y, Z_i)$$

showing that $X \otimes Y$ is compact. Parts (2) and (3) are similar and are left as Exercise A.5. ☐

The following is then immediate:

Corollary 2.32. *For any tensor-triangulated category T, the full subcategory of rigid objects forms a tensor-triangulated category.* ☐

3. Geometric functors

In this section we introduce the notion of a *geometric functor*. In order to demonstrate their powerful formal properties, we recall the adjoint functor theorems for triangulated categories.

Lemma 3.1. *Let $F\colon \mathsf{T} \to \mathsf{U}$ be a triangulated functor. If F has a adjoint (on either side), then this adjoint is also a triangulated functor.*

Proof. Left as Exercise A.8. ☐

Theorem 3.2 (Adjoint functor theorems). *Let T and U be triangulated categories. Let $F\colon \mathsf{T} \to \mathsf{U}$ be a triangulated functor, and suppose that T is compactly generated.*

(1) F is coproduct preserving if and only if F has a triangulated right adjoint.
(2) F is product preserving if and only if F has a triangulated left adjoint.

Proof. This is an immediate consequence of Brown representability and Lemma 3.1. □

Proposition 3.3. *Let $F : \mathsf{T} \rightleftarrows \mathsf{U} : G$ be an adjunction between triangulated categories.*

(1) If G preserves coproducts, then F preserves compact objects.
(2) Suppose that T is compactly generated and that F preserves compacts. Then G preserves coproducts.

Proof. For (1) we let $C \in \mathsf{T}^c$. For any set of objects $\{X_i\}$ in U, we have isomorphisms

$$\oplus \mathrm{Hom}(FC, X_i) \simeq \oplus \mathrm{Hom}(C, GX_i) \simeq \mathrm{Hom}(C, \oplus GX_i) \simeq \mathrm{Hom}(C, G(\oplus X_i)) \simeq \mathrm{Hom}(FC, \oplus X_i)$$

using in turn, the adjunction, compactness of C, the assumption that G preserves coproducts, and finally the adjunction again. For (2), by compact generation of T it suffices to prove that the canonical map $\oplus GX_i \to G(\oplus X_i)$ induces isomorphisms

$$\mathrm{Hom}(C, \oplus GX_i) \xrightarrow{\simeq} \mathrm{Hom}(C, G(\oplus X_i))$$

for all $C \in \mathsf{T}^c$. This is then a straightforward check using the adjunction and the fact that F preserves compacts. □

We now have the necessary triangulated background to make the key definition and prove some key features of it. We will then record several examples afterwards.

Definition 3.4. Let T and U be rigidly-compactly generated tt-categories. A *geometric functor* $f^*\colon \mathsf{T} \to \mathsf{U}$ is a strong symmetric monoidal, triangulated functor which preserves coproducts.

Proposition 3.5. *Let $f^*\colon \mathsf{T} \to \mathsf{U}$ be a geometric functor. Then f^* fits into an adjoint triple*

$$\begin{tikzcd} \mathsf{T} \arrow[r, shift left=2, "f^*"] \arrow[r, shift right=2, "f^{(1)}"'] & \mathsf{U} \arrow[l, "f_*" description] \end{tikzcd}$$

where left adjoints are denoted on top.

Proof. Since f^* is coproduct preserving, it has a right adjoint f_* by Theorem 3.2. Since f^* is a strong monoidal functor it preserves rigid objects (Lemma 2.23) and hence preserves compact objects since these coincide by assumption. Therefore, f_* preserves coproducts by Proposition 3.3 and hence itself has a right adjoint $f^{(1)}$ again by Theorem 3.2. □

Let $f^*\colon \mathsf{T} \to \mathsf{U}$ be a geometric functor. There is a natural map $\pi\colon X \otimes f_*Y \to f_*(f^*X \otimes Y)$ for all $X \in \mathsf{T}$ and $Y \in \mathsf{U}$ which is constructed as the adjunct to the natural map

$$f^*(X \otimes f_*Y) \simeq f^*X \otimes f^*f_*Y \xrightarrow{f^*X \otimes \varepsilon} f^*X \otimes Y$$

where ε is the counit of the (f^*, f_*) adjunction.

Proposition 3.6 (The projection formula). *Let $f^*\colon \mathsf{T} \to \mathsf{U}$ be a geometric functor. The natural map*

$$\pi\colon X \otimes f_*Y \xrightarrow{\simeq} f_*(f^*X \otimes Y)$$

is an isomorphism for all $X \in \mathsf{T}$ and $Y \in \mathsf{U}$.

Proof. Let $Z \in \mathsf{T}$ and $Y \in \mathsf{U}$ be arbitrary. We first prove that π is an isomorphism if X is rigid. In this case, we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}(Z, X \otimes f_*Y) &\simeq \mathrm{Hom}(Z \otimes DX, f_*Y) && \text{since } X \text{ is rigid} \\
&\simeq \mathrm{Hom}(f^*Z \otimes f^*(DX), Y) && \text{by adjunction and monoidality of } f^* \\
&\simeq \mathrm{Hom}(f^*Z \otimes D(f^*X), Y) && \text{by Lemma 2.23} \\
&\simeq \mathrm{Hom}(Z, f_*(f^*X \otimes Y)) && \text{by rigidity of } D(f^*X) \text{ and adjunction.}
\end{aligned}$$

A diagram chase verifies that this isomorphism is induced by π (we omit tedious manipulations of commutative diagrams throughout in this course). By the Yoneda lemma we therefore deduce that π is an isomorphism when X is rigid. Now the collection of $X \in \mathsf{T}$ for which π is an isomorphism (for a fixed Y) is a localising subcategory (since both f^* and f_* preserve coproducts and are triangulated). This localising subcategory contains all the rigid=compact objects, and hence by compact generation, we deduce that π is an isomorphism for all X. $\square$

A slick trick allows one to generate new isomorphisms from these by using uniqueness of adjunctions. Recall that if $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $F_1F_2 \dashv G_2G_1$ (notice the order reversal).

Proposition 3.7. *Let $f^*\colon \mathsf{T} \to \mathsf{U}$ be a geometric functor. There are natural isomorphisms:*

$$\underline{\mathrm{Hom}}(X, f_*Y) \simeq f_*\underline{\mathrm{Hom}}(f^*X, Y) \tag{3.8}$$

$$\underline{\mathrm{Hom}}(f_*X, Y) \simeq f_*\underline{\mathrm{Hom}}(X, f^{(1)}Y) \tag{3.9}$$

$$f^{(1)}\underline{\mathrm{Hom}}(X, Y) \simeq \underline{\mathrm{Hom}}(f^*X, f^{(1)}Y). \tag{3.10}$$

Proof. We first fix X in the projection formula (Proposition 3.6). We have adjunctions:

$$X \otimes f_*(-) = (X \otimes -) \circ f_* \dashv f^{(1)} \circ \underline{\mathrm{Hom}}(X, -)$$

and

$$f_*(f^*X \otimes -) \dashv \underline{\mathrm{Hom}}(f^*X, -) \circ f^{(1)}.$$

By uniqueness of adjoints, we therefore deduce that $f^{(1)}\underline{\mathrm{Hom}}(X, Y) \simeq \underline{\mathrm{Hom}}(f^*X, f^{(1)}Y)$, giving (3.10). Fixing Y in the projection formula instead yields (3.9). Finally, for (3.8), we have an isomorphism $f^*X \otimes f^*(-) \simeq f^*(X \otimes -)$. Taking right adjoints of this gives the claim. $\square$

Let us return to the examples and see what the above features of geometric functors correspond to in this setting.

Example 3.11. Let $f\colon R \to S$ be a map of commutative rings. Then the extension of scalars functors $f^* = S \otimes^{\mathsf{L}}_R -\colon \mathsf{D}(R) \to \mathsf{D}(S)$ is a geometric functor. Its right adjoint f_* is the restriction of scalars functor, and the double right adjoint is the coextension of scalars functor $f^{(1)} = \mathsf{R}\mathrm{Hom}_R(S, -)$. In this case, the projection formula is not exciting: it says that for $X \in \mathsf{D}(R)$ and $Y \in \mathsf{D}(S)$ there is an isomorphism $X \otimes^{\mathsf{L}}_R Y \simeq (S \otimes^{\mathsf{L}}_R X) \otimes^{\mathsf{L}}_S Y$. The other isomorphisms are similarly "obvious" in this setting. The main selling point of the framework is that these are general isomorphisms which hold in all settings, and moreover admit uniform proofs based purely on formal manipulations.

Example 3.12. Let G be a group and H be a subgroup. There is a restriction functor $f^* = \mathrm{res}^G_H\colon \mathsf{StMod}(kG) \to \mathsf{StMod}(kH)$ given by restricting the G-action to a H-action, and this is a geometric functor. This has a right adjoint given by coinduction $f_* = \mathrm{Hom}_{kH}(kG, -)$ which has a further right adjoint given by restriction again. In this example, we see some extra features happening: f^* also has a left adjoint (the induction $kG \otimes_{kH} -$) and this left adjoint is isomorphic to the coinduction. Therefore we obtain an infinite tower of adjunctions. This is an instance of the *Wirthmüller isomorphism* which we will see later on.

Remark 3.13. We see that the extra features visible in Example 3.12 are not uniform for all geometric functors. Indeed, they do not occur in the setting of Example 3.11 in general. However, one can completely characterise the existence of extra adjoints and when these are isomorphic in general tt-terms. This is the point of much of the rest of this course.

4. Grothendieck–Neeman duality

5. Grothendieck duality via proxy-smallness

6. The Wirthmüller isomorphism

7. Matlis duality

8. Orientability and Gorenstein duality

Appendix A. Exercises

Exercise A.1. Prove Proposition 2.11.

Exercise A.2. Prove that for any ring R, the compact objects in $\mathsf{D}(R)$ are the perfect complexes.

Exercise A.3. Prove that if $M \in \mathsf{stmod}kG$ (that is, M is finitely generated), then M is compact in $\mathsf{StMod}(kG)$.

Exercise A.4. Let T be a tt-category and let $Y \in \mathsf{T}$ be rigid. Prove that the natural map

$$F(X, \mathbb{1}) \otimes Y \to F(X, Y)$$

is an isomorphism for all $X \in \mathsf{T}$.

Exercise A.5. Let T be a tensor-triangulated category and $X \in \mathsf{T}$ be rigid. Prove that if Y is internally compact (resp., rigid), then $X \otimes Y$ is internally compact (resp., rigid). (See Proposition 2.31.)

Exercise A.6. Let $F\colon \mathsf{T} \to \mathsf{U}$ be a tt-functor between tt-categories. Let $X \in \mathsf{T}$ be rigid, and $Y \in \mathsf{T}$ be arbitrary. Prove that there is a natural isomorphism $F\underline{\mathrm{Hom}}(X, Y) \simeq \underline{\mathrm{Hom}}(FX, FY)$.

Exercise A.7. Let $\mathcal{K}$ be a set of compact objects in a rigidly-compactly generated tt-category T. Consider the localising $\otimes$-ideal $\mathsf{L} = \mathrm{Loc}_{\otimes}(\mathcal{K})$. Explain how one can view this as a tt-category (note that it is *not* a tt-subcategory of T). Which of the following are true in general?:

(1) if X is compact in L then X is compact in T;
(2) if $X \in \mathsf{L}$ is compact in T then X is compact in L;
(3) if X is rigid in L then X is rigid in T;
(4) if $X \in \mathsf{L}$ is rigid in T then X is rigid in L;
(5) $\{$compacts in $\mathsf{L}\} \subset \{$rigids in $\mathsf{L}\}$;
(6) $\{$compacts in $\mathsf{L}\} \supset \{$rigids in $\mathsf{L}\}$;
(7) $\{$compacts in $\mathsf{L}\} = \{$rigids in $\mathsf{L}\}$.

Exercise A.8. Prove Lemma 3.1.