DUALITIES IN TRIANGULATED CATEGORIES

Overview. The goal of this lecture course is to give a modern point of view on some important duality theorems in algebra, from the point of view of triangulated categories. This perspective also enables one to view these dualities not just in an algebraic setting, but to transport them into other realms, such as geometry and topology. The main focus will be on Grothendieck's local duality theorem, which relates the Matlis dual of local cohomology to the ordinary functional dual. The course will give an introduction to triangulated categories, before turning to introducing local cohomology, firstly in the classical algebraic setting, and then in the triangulated realm and explaining how the latter recovers and generalises the former. We will then turn to exploring local duality in the triangulated setting, which naturally leads us to consider other duality theorems such as Greenlees-May duality, and Warwick duality. We will show how one can recover the classical statement of Grothendieck local duality from this more general triangulated version.

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Relevant literature.

(1) H. Krause. Localization theory for triangulated categories. In Triangulated categories, volume 375 of London Math. Soc. Lecture Note Ser., pages 161–235. Cambridge Univ. Press, Cambridge, 2010
(2) H. Krause. Homological theory of representations, volume 195 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2022
(3) M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. Mem. Amer. Math. Soc., 128(610):x+114, 1997
(4) T. Barthel, D. Heard, and G. Valenzuela. Local duality in algebra and topology. Adv. Math., 335:563–663, 2018
(5) W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. Amer. J. Math., 124(1):199–220, 2002
(6) C. Huneke. Lectures on local cohomology. Contemp. Math., 436, Interactions between homotopy theory and algebra, 51–99, Amer. Math. Soc., Providence, RI, 2007.
(7) C. Weibel. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics. 38. Cambridge: Cambridge University Press. xiv, 450 p. (1994).
(8) A. Neeman. Triangulated categories. Annals of Mathematics Studies. 148. Princeton, NJ: Princeton University Press. vii, 449 p. (2001).

Assessment. The final exam will be an oral exam. For zápočet, students will have to get at least 50% of marks on each of the 3 homework assignments.

1. What is duality?

There is no precise definition of what constitutes a duality, indeed, Atiyah said "Duality in mathematics is not a theorem, but a "principle"". Perhaps the closest to a precise formulation of duality is that it is a contravariant endofunctor $D\colon \mathcal{C} \to \mathcal{C}$ such that D^2 is the identity, either on $\mathcal{C}$ or on a convenient subcategory of it. However there are many forms of things which we call duality theorems which do not fit this mould; for example, there is even something called the covariant Grothendieck duality theorem! Another possible formulation of a duality is that it is a statement which relates a *co*variant functor to a *contra*variant functor. Let us investigate three examples of dualities to give a flavour of the meaning.

Example 1.1 (Complements of subsets)**.** Let A be a set, and $B \subseteq A$. Then taking the complement twice, we have $(B^c)^c = B$. In the above formulation, this amounts to taking $\mathcal{C}$ to be the category whose objects are subsets of A, with a morphism $B \to B'$ if and only if $B \subseteq B'$, and $D = (-)^c$.

Example 1.2 (Functional duality of vector spaces)**.** Let V be a vector space over a field k. The dual vector space V^* is defined to be the set of k-linear maps $V \to k$ with the obvious vector space structure, i.e., $V^* = \mathrm{Hom}_k(V, k)$. There is a natural map $f\colon V \to V^{**}$ defined via

$$f(v)\colon \mathrm{Hom}_k(V,k) \to k \qquad f(v)(g) = g(v).$$

One checks that this map is k-linear, and that moreover, if V is finite dimensional, then f is an isomorphism. Rephrasing this categorically, we take $\mathcal{C}$ to be the category of k-vector spaces, $D = (-)^*$, and D^2 is isomorphic to the identity on the subcategory of finite dimensional vector spaces. There is an important note to be made here: as finite dimensional vector spaces are determined by their dimension, one may check that for V finite dimensional, we have that V is isomorphic to V^*. However, this isomorphism is *not* natural since it relies on a choice of basis. On the other hand, V is naturally/canonically equivalent to its double dual V^{**}. As such, in the prototype definition of duality given above, we actually want to require that D^2 is *naturally* isomorphic to the identity.

Example 1.3 (Grothendieck local duality)**.** We now turn to stating the main duality theorem of this course. We will not define all of the terms in the statement; we will make them precise throughout the course. We will focus on the statement in commutative algebra, but one selling point of the language which we will study in this course, is that it allows for a statement of Grothendieck local duality to be made in a broad range of settings.

Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring; recall that a ring is Gorenstein if it has finite injective dimension as a module over itself. Then Grothendieck local duality asserts that

$$\mathrm{Ext}^i_R(M, R^\wedge_\mathfrak{m}) = H^{\dim(R)-i}_\mathfrak{m}(M)^\vee$$

for all R-modules M.

In this statement, $H^*_\mathfrak{m}(-)$ denotes the *local cohomology.* This is a much used tool in commutative algebra and beyond. One example of where local cohomology can be used is in answering questions about how many generators one needs to generate an ideal up to radical. Recall that for an ideal I, the radical is

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n\}.$$

For example, in the polynomial ring $k[x,y]$ the ideal $I = (x^2, xy, y^2)$ can actually be generated up to radical by only two elements; that is, $\sqrt{I} = \sqrt{(x^2, y^2)}$. This example is very small, but for larger rings and ideals, local cohomology provides a structured way to attack such questions.

Grothendieck local duality does not fit the mould for duality theorems as we 'defined' above. However, note that it deserves the title of a duality since it relates the contravariant functor $\operatorname{Ext}^i_R(-, R^\wedge_\mathfrak{m})$ to the covariant functor $H^{\dim(R)-i}_\mathfrak{m}(-)$.

Grothendieck local duality is a powerful tool since it enables one to replace questions about local cohomology with questions about Ext-groups. Another reason Grothendieck local duality is useful arises when trying to pass to local problems: $H^*_\mathfrak{m}(-)_\mathfrak{p}$ is always zero if $\mathfrak{p} \neq \mathfrak{m}$, so one cannot simply localize local cohomology directly. Instead one may pass to the world of Ext-groups, localize there, and then translate back to local cohomology.

2. Triangulated categories

2.1. **The axioms.** Loosely speaking, a triangulated category consists of an additive category T, together with two extra pieces of data:

1. an equivalence of categories $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$ called the *shift*;
2. a collection of *triangles* $X \to Y \to Z \to \Sigma X$ satisfying various axioms which ensure good behaviour.

Definition 2.1. Let T be an additive category and $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$ be an additive equivalence of categories. A *candidate triangle* is a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

such that the composites $g \circ f$, $h \circ g$, and $\Sigma f \circ h$ are all zero. A morphism of candidate triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'. \end{array}$$

Considering a smaller class of candidate triangles which satisfy certain properties leads to the notion of a triangulated category. The first four axioms are easy to justify, but the final axiom is harder to motivate. It is convenient to develop the theory assuming only these first four axioms, and then add in the final one once it becomes relevant. Nonetheless, we'll give both definitions now, so that we can consider an example before embarking on the abstract theory.

Definition 2.2. A *pretriangulated category* T is an additive category together with an additive equivalence of categories $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$, and a subclass of candidate triangles called *distinguished triangles* which satisfy the following axioms:

(TR0) Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle, and for all $X \in \mathsf{T}$ the candidate triangle

$$X \xrightarrow{1} X \to 0 \to \Sigma X$$

is distinguished.

(TR1) For all $f\colon X \to Y$ in T, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$.

(TR2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a candidate triangle. This is distinguished if and only if the candidate triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished.

(TR3) For any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'. \end{array}$$

in which the rows are triangles, there exists a map $w \colon Z \to Z'$ (which need *not* be unique) making

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'. \end{array}$$

commute.

Remark 2.3. It is standard to drop the adjective 'distinguished', and just refer to them as *triangles*. For candidate triangles, we will never drop the adjective.

Remark 2.4. By combining (TR2) with (TR3), one sees that there always exists fillers in the first and second column too.

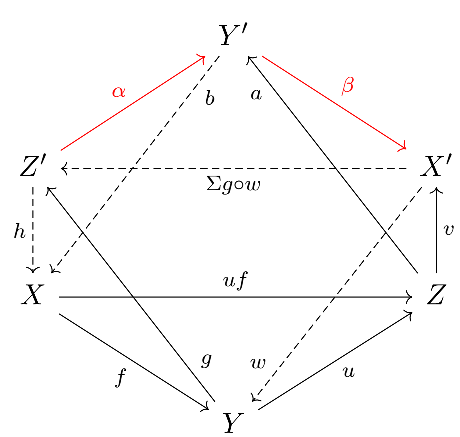
Definition 2.5. A *triangulated category* T is a pretriangulated category satisfying the following additional axiom:

(TR4) Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z' \xrightarrow{h} \Sigma X$, $Y \xrightarrow{u} Z \xrightarrow{v} X' \xrightarrow{w} \Sigma Y$ and $X \xrightarrow{uf} Z \xrightarrow{a} Y' \xrightarrow{b} \Sigma X$ are distinguished triangles. Then there exists a distinguished triangle

$$Z' \xrightarrow{\alpha} Y' \xrightarrow{\beta} X' \xrightarrow{\gamma} \Sigma Z'$$

such that $v = \beta a$, $h = b\alpha$, $\gamma = \Sigma g \circ w$, $w\beta = \Sigma f \circ b$ and $\alpha g = au$.

Pictorially this axiom can be represented by the following commuting diagram:



The dotted maps are of degree 1 (i.e., $f\colon X \dashrightarrow Y$ represents a map $f\colon X \to \Sigma Y$), and composites of the form $\to\to\dashrightarrow$ are triangles. The red maps are the extra data, together with the condition that they form a triangle. In order to remember this, note that the primed letters are the cones of maps, and that every triangle contains an X, Y, and a Z (primed or otherwise). In light of the shape of the above diagram, (TR4) is often referred to as the *octahedral axiom*.

Alternatively, one can give the following pictorial representation.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z' & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{uf} & Z & \xrightarrow{a} & Y' & \xrightarrow{b} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma f} \\
Y & \xrightarrow{u} & Z & \xrightarrow{v} & X' & \xrightarrow{w} & \Sigma Y \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Sigma g} \\
Z' & \overset{\alpha}{\dashrightarrow} & Y' & \overset{\beta}{\dashrightarrow} & X' & \overset{\gamma}{\dashrightarrow} & \Sigma Z'
\end{array}$$

The first three rows are the given triangles, and (TR4) then asserts the existence of the dotted arrows making the diagram commute, so that the bottom row is also a triangle.

Let's briefly discuss the axioms and provide some motivation for them. If one thinks as triangles as a generalisation of short exact sequences, then in a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ you should think of Z as the (homotopy coherent) cokernel of f, and X as the (homotopy coherent) kernel of g. The axioms then mean the following.

(TR0) The kernel and cokernel of the identity is zero.
(TR1) Every map has a kernel and cokernel.
(TR2) Up to sign, every map is the kernel of its cokernel and vice versa.
(TR3) Kernels and cokernels are almost functorial.
(TR4) One can interpret the given triangles as saying $Z' \simeq Y/X$, $X' \simeq Z/Y$ and $Y' \simeq Z/X$, and then the axiom asserts that $X' \simeq Y'/Z'$, i.e., $(Z/X)/(Y/X) \simeq Z/Y$.

Remark 2.6. Given a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$, it is common to call Z the *cofibre* (or *cone*) of f, and X the *fibre* (or *cocone*) of g. Sometimes it is customary to write triangles as $X \to Y \to Z$ and drop the map to the shift. We will sometimes subscribe to this later on the course for brevity, but we warn the reader that it is important not to forget this map. For example, a morphism of triangles requires the square including the map $Z \to \Sigma X$ to commute.

We will give two detailed examples of triangulated categories: the homotopy category of a ring and the derived category of a ring. These are intrinsically algebraic examples; there are many more examples of triangulated categories, including plenty coming from topology and geometry, but these require more background to describe, so we will focus on these algebraic cases. Here's a list of some examples we won't describe, included for the interested reader:

- the stable homotopy category, and equivariant and chromatic versions of this,
- the derived category of quasi-coherent sheaves over a nice enough scheme,
- various 'mixed' versions of these, such as the motivic stable homotopy category,
- the stable module category of a finite group.

2.2. **Aside: preliminaries from homological algebra.** Homological algebra is built upon taking resolutions of modules. Therefore, one seeks a category which contains precisely the homological information of modules, so that objects are resolutions, and a module is isomorphic to any resolution of it. This utopian category will the derived category of the ring in question.

Let R be a ring. A chain complex of R-modules M is a collection of R-modules $\{M_i\}_{i\in\mathbb{Z}}$ together with maps called differentials, $d_i\colon M_i \to M_{i-1}$ for all $i \in \mathbb{Z}$, satisfying $d_i \circ d_{i+1} = 0$. We write $\mathrm{Ch}(R)$ for the category of chain complexes.

Recall that the condition on the differential ensures that $\mathrm{Im}(d_{i+1}) \subseteq \mathrm{Ker}(d_i)$, so that we may consider the homology groups $H_i(M) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1})$ which measure how far away from being exact a sequence is. A map $f\colon M \to N$ of chain complexes is a collection of levelwise maps $f_i\colon M_i \to N_i$ for each $i \in \mathbb{Z}$ which commute with the differentials. Such a map is often called a chain map. A chain map $f\colon M \to N$ is said to be a *quasi-isomorphism* if

$$H_i(f)\colon H_iM \to H_iN$$

is an isomorphism for all $i \in \mathbb{Z}$. Recall that given any short exact sequence of complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

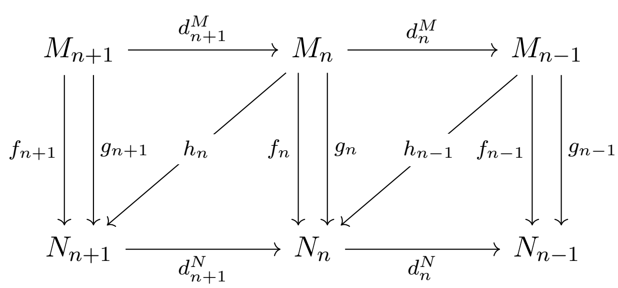
there is a connecting map $\delta\colon H_nC \to H_{n-1}A$ so that the induced sequence

$$\cdots \to H_nB \to H_nC \to H_{n-1}A \to H_{n-1}B \to \cdots$$

is long exact.

In this language, one may rephrase the definition of projective resolution of a module M as being a complex P consisting of projective modules, together with a chain map $P \to M[0]$ which is a quasi-isomorphism. Note here that we view the *module* M, as a *complex* $M[0]$ by putting M in degree 0 and zeroes everywhere else. (Henceforth we will just write M for $M[0]$ and leave it implicit that modules are viewed as complexes in degree 0.)

Given chain maps $f, g\colon M \to N$, a *chain homotopy* from f to g is a collection of maps $h_n\colon M_n \to N_{n+1}$ such that $f_n - g_n = d^N_{n+1}h_n + h_{n-1}d^M_n$ as demonstrated by the diagram



We say that two complexes M, N are *chain homotopy equivalent* if there exists maps $f\colon M \to N$ and $g\colon N \to M$ so that gf is chain homotopic to the identity on M and fg is chain homotopic to the identity on N.

Let $f\colon M \to N$ be a map of chain complexes. The mapping cone of f denoted $C(f)$ is the complex with $C(f)_n = M_{n-1} \oplus N_n$ and differential $d(m, n) = (-dm, dn - fm)$. Sometimes it can be convenient to write this differential as the matrix

$$\begin{pmatrix} -d_M & 0 \\ -f & d_N \end{pmatrix}.$$

The *suspension* (or *shift*) of a complex M, denoted ΣM, is defined by $(\Sigma M)_n = M_{n-1}$ with differential $-d$. A way to remember which way the shift moves, is to note that the shift is opposite to the differential. By including the M factor and sending the N factor to 0 one obtains a chain map $c_f\colon C(f) \to \Sigma M$, $c_f(m,n) = m$, as demonstrated in the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_n \oplus N_{n+1} & \xrightarrow{(-d_M,\ d_N - f)} & M_{n-1} \oplus N_n & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \\ \cdots & \longrightarrow & M_n & \xrightarrow[-d_M]{} & M_{n-1} & \longrightarrow & \cdots \end{array}$$

Similarly, there is a chain map $i_f\colon N \to C(f)$ given by $i_f(n) = (0,n)$.

The tensor product of chain complexes is defined by

$$(M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j$$

with differential $d(m \otimes n) = (dm \otimes n) + (-1)^{|m|}(m \otimes dn)$. The internal hom of chain complexes is defined by

$$\mathrm{Hom}_R(M,N)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$$

with differential $d(f) = d^N \circ f - (-1)^{|f|} f \circ d^M$. If R is a commutative ring, then for any complex of R-modules M, the tensor product functor $- \otimes_R M\colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is left adjoint to the internal hom functor $\mathrm{Hom}_R(M,-)$.

2.3. **Example: the homotopy category of a ring.** Fix a ring R. Before we can define our desired category in which quasi-isomorphisms are inverted, it is convenient to introduce a stepping stone towards this category, called the homotopy category of complexes. In this category we kill the null homotopic maps. Recall that a map $f\colon M \to N$ is null homotopic if it is chain homotopic to the zero map. We write $\mathrm{Null}(M,N)$ for the subgroup of $\mathrm{Hom}_{\mathrm{Ch}(R)}(M,N)$ consisting of the null homotopic maps.

The *homotopy category* $\mathsf{K}(R)$ is defined by having objects the chain complexes of R-modules, and morphisms given by the homotopy classes of chain maps, i.e.,

$$\mathrm{Hom}_{\mathsf{K}(R)}(M,N) = \mathrm{Hom}_{\mathrm{Ch}(R)}(M,N)/\mathrm{Null}(M,N).$$

Equivalently, the morphisms are the chain maps up to the equivalence relation of chain homotopy equivalence. The distinguished triangles in the homotopy category are the triangles which are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopic) to those of the form

$$M \xrightarrow{f} N \to C(f) \to \Sigma M$$

for some map of chain complexes $f\colon M \to N$.

Let us now prove that $\mathsf{K}(R)$ is a triangulated category. Due to the number of axioms, this is a bit of a slog, but it is worthwhile seeing the details spelled out.

Theorem 2.7. *Let R be a ring. The category $\mathsf{K}(R)$ with the triangles those which are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopic) to those of the form*

$$M \xrightarrow{f} N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M$$

for some map of chain complexes $f\colon M \to N$ is a triangulated category.

Proof of (TR0). For (TR0) it suffices to prove that $C(1_M)$ is null homotopic, in other words, the identity map on M is null homotopic for all M. Recall that $C(1_M)_n = M_n \oplus M_{n+1}$ with differential $(-d, d-1)$. Define $h_n\colon M_n \oplus M_{n+1} \to M_{n+1} \oplus M_{n+2}$ by $h_n(x,y) = (-y, 0)$. One easily verifies that this defines a chain homotopy from the identity on the cone of the identity to the zero map. Hence $C(1_M)$ is null homotopic. □

Proof of (TR1). This is immediate from the definition of the triangles in $\mathsf{K}(R)$. □

Proof of (TR2). Consider the triangle

$$M \xrightarrow{f} N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M.$$

We must show that the candidate triangle

$$N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M \xrightarrow{-\Sigma f} \Sigma N$$

is also a triangle. (The argument for rotating the other direction is analogous so we omit it.) So we show that taking the cone of i yields a triangle which is isomorphic to this candidate triangle.

Firstly we verify that $C(i)$ and ΣM are isomorphic in $\mathsf{K}(R)$ (i.e., chain homotopy equivalent). We will then show that this is compatible with the triangles. Recall that $i\colon N \to C(f)$ is defined by $n \mapsto (0, n)$. Therefore, $C(i)_n = N_{n-1} \oplus C(f)_n = N_{n-1} \oplus M_{n-1} \oplus N_n$ with differential

$$\begin{pmatrix} -d_N & 0 \\ -i & d_{C(f)} \end{pmatrix} = \begin{pmatrix} -d_N & 0 & 0 \\ 0 & -d_M & 0 \\ -\mathrm{id}_N & -f & d_N \end{pmatrix}.$$

Firstly we define a map $\alpha\colon \Sigma M \to C(i)$ by $\alpha(m) = (-f(m), m, 0)$. To verify this is a chain map we must show that

$$\begin{array}{ccc} M_{n-1} & \xrightarrow{\quad -d \quad} & M_{n-2} \\ {\scriptstyle \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix}} \\ C(i)_n & \xrightarrow[\begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ -1 & -f & d \end{pmatrix}]{} & C(i)_{n-1} \end{array}$$

commutes. One calculates that

$$\begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ -1 & -f & d \end{pmatrix} \begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} df \\ -d \\ 0 \end{pmatrix}$$

and therefore the square commutes so that α is indeed a chain map. We also define a map $\beta\colon C(i) \to \Sigma M$ by $(n, m, n') \mapsto m$ (i.e., the matrix $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$). One easily verifies that this is a chain map.

The composite $\beta\alpha$ is the identity since $\beta(\alpha(m)) = \beta(-f(m), m, 0) = m$. On the other hand, $\alpha\beta(n, m, n') = \alpha(m) = (-f(m), m, 0)$ so $\alpha\beta$ is not the identity. However, let us show that it is indeed chain homotopic to the identity. Define $h_j\colon C(i)_j \to C(i)_{j+1}$ by $h_j(n, m, n') = (n', 0, 0)$.

One then calculates that $dh + hd = \alpha\beta - 1$ so that $\alpha\beta$ is chain homotopic to the identity. Therefore $\alpha\colon \Sigma M \to C(i)$ is an isomorphism in $\mathsf{K}(R)$. Therefore it only remains to verify that the diagram

$$\begin{array}{ccccccc}
N & \xrightarrow{i} & C(f) & \xrightarrow{c} & \Sigma M & \xrightarrow{-\Sigma f} & \Sigma N \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} \\
N & \xrightarrow[i]{} & C(f) & \xrightarrow[\text{inc}]{} & C(i) & \xrightarrow[c']{} & \Sigma N
\end{array}$$

commutes in $\mathsf{K}(R)$ since then the top row is isomorphic (as a candidate triangle) to the bottom row which is a triangle. Hence it is by definition also a triangle as required.

The left hand square clearly commutes on the nose. The right most square also commutes on the nose, since $c'\alpha(m) = c'(-f(m), m, 0) = -f(m)$. For the middle square, $\alpha c(m, n') = \alpha(m) = (-f(m), m, 0)$ whereas $\text{inc}(m, n') = (0, m, n')$. Therefore the middle square does not commute in the category of chain complexes, but we will show that it does commute in $\mathsf{K}(R)$. Since $\alpha\beta$ is the identity in $\mathsf{K}(R)$ (as proved above), it suffices to verify that $\beta \circ \text{inc} = c$ instead, which is easy from the definitions. This completes the proof of (TR2). □

Proof of (TR3). By definition of the triangles, we may assume that we are given the following solid diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{f} & N & \xrightarrow{i_f} & C(f) & \xrightarrow{c_f} & \Sigma X \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\
M' & \xrightarrow[g]{} & N' & \xrightarrow[i_g]{} & C(g) & \xrightarrow[c_g]{} & \Sigma X'.
\end{array}$$

in which the first square commutes, and we must define the dotted map w. The first square commuting, means that vf and gu are chain homotopic. Therefore, for each n there exists maps $h_n\colon N_n \to M'_{n+1}$ such that $gu - vf = dh + hd$. Define a map $w\colon C(f) \to C(g)$ by the matrix

$$\begin{pmatrix} u & 0 \\ h & v \end{pmatrix}.$$

We need to check that this is indeed a chain map, and that it makes both squares commute in the diagram. These are easy calculations which are left to the reader. □

Proof of (TR4). We may assume that the triangles are 'standard' ones, so we assume the existence of a commuting diagram as follows, whose rows are triangles.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i_f} & C(f) & \xrightarrow{c_f} & \Sigma X \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u} & & \vdots{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{uf} & Z & \xrightarrow{i_{uf}} & C(uf) & \xrightarrow{c_{uf}} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \vdots{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma f} \\
Y & \xrightarrow{u} & Z & \xrightarrow{i_g} & C(u) & \xrightarrow{c_u} & \Sigma Y \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Sigma g} \\
C(f) & \overset{\alpha}{\dashrightarrow} & C(uf) & \overset{\beta}{\dashrightarrow} & C(u) & \overset{\gamma}{\dashrightarrow} & \Sigma C(f)
\end{array}$$

We must construct dotted maps so that the bottow row is also a triangle. By functoriality of the mapping cone as proved in (TR3), we have maps $\alpha\colon C(f) \to C(uf)$ and $\beta\colon C(uf) \to C(u)$ making the whole diagram commute. We define $\gamma\colon C(u) \to \Sigma C(f)$ (note that this equates to maps $Y_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1}$) by $(y, z) \mapsto (0, y)$. This clearly makes the bottom right square commute also.

Therefore it remains to verify that $C(f) \xrightarrow{\alpha} C(uf) \xrightarrow{\beta} C(u) \xrightarrow{\gamma} \Sigma C(f)$ is a triangle. Define $w\colon C(\alpha) \to C(u)$ by $w(x, y, x', z) = (y + f(x'), z)$, and define $\widetilde{w}\colon C(u) \to C(\alpha)$ by $\widetilde{w}(y, z) = (0, y, 0, z)$. Consider the diagrams

$$\begin{array}{ccc} C(uf) & \xrightarrow{\beta} & C(u) \\ {\scriptstyle 1}\downarrow & & \uparrow{\scriptstyle w} \\ C(uf) & \xrightarrow{i_\alpha} & C(\alpha) \end{array} \qquad \begin{array}{ccc} C(u) & \xrightarrow{\gamma} & \Sigma C(f) \\ {\scriptstyle \widetilde{w}}\downarrow & & \downarrow{\scriptstyle 1} \\ C(\alpha) & \xrightarrow{c_\alpha} & \Sigma C(f) \end{array}$$

where $\beta(x, z) = (fx, z)$ (as in the proof of (TR3)). One easily checks from the definitions that both of these diagrams commute. Therefore it only remains to prove that w is a chain homotopy equivalence. The composite $w\widetilde{w}$ is equal to the identity. For the other composite, define

$$h_n\colon C(\alpha)_n = X_{n-1} \oplus Y_n \oplus X_n \oplus X_{n+1} \to C(\alpha)_{n+1} = X_n \oplus Y_{n+1} \oplus X_{n+1} \oplus X_{n+2}$$

by $h_n(x, y, x', z) = (x', 0, 0, 0)$. We leave it to the reader to check that this defines a chain homotopy showing that $\widetilde{w}w$ is homotopic to the identity. □

We end our discussion of the homotopy category of a ring, by proving it has a universal property. There is a functor $h\colon \mathrm{Ch}(R) \to \mathsf{K}(R)$ defined to be the identity on objects, and to send a map f to its equivalence class $[f]$ under the relation of chain homotopy.

Proposition 2.8. *Let $F\colon \mathrm{Ch}(R) \to \mathcal{C}$ be a functor which is homotopy invariant, i.e., if $f \simeq g$ then $F(f) = F(g)$. Then there exists a unique functor $\bar{F}\colon \mathsf{K}(R) \to \mathcal{C}$ making the diagram*

$$\begin{array}{ccc} \mathrm{Ch}(R) & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle h}\downarrow & \nearrow{\scriptstyle \bar{F}} & \\ \mathsf{K}(R) & & \end{array}$$

commute.

Proof. Uniqueness is immediate from the fact that h is the identity on objects, and is full. For existence, define $\bar{F}(M) = F(M)$ on objects, and $\bar{F}([f]) = F(f)$ on maps. We note that this is well-defined since F is homotopy invariant by assumption. □

We'd next like to define a category in which the quasi-isomorphisms are inverted. Before we can do this, we establish some more basic properties of general triangulated categories.

2.4. **Basic properties of triangulated categories.** A nice tool in triangulated categories which we will use throughout this course, is a version of the 5 lemma which we will prove in this section.

Definition 2.9. Let T be a pretriangulated category and A be an abelian category. An additive functor $H\colon \mathsf{T} \to \mathsf{A}$ is *homological* if for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, the induced sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

is exact. Such a functor which is contravariant is said to be *cohomological*.

The following lemma shows that one can also rephrase the definition of homological to be those additive functors which turn triangles into long exact sequences.

Lemma 2.10. *If $H\colon \mathsf{T} \to \mathsf{A}$ is a homological functor, then for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, applying H yields a long exact sequence*

$$\cdots \to H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}w)} H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \to \cdots$$

Proof. By (TR2), $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is also a triangle, so

$$HY \xrightarrow{H(g)} HZ \xrightarrow{H(h)} H(\Sigma X)$$

is exact. Repeating this procedure gives the claim. □

Lemma 2.11. *Let T be a pretriangulated category, and $A \in \mathsf{T}$. The functor $\mathrm{Hom}_{\mathsf{T}}(A,-)\colon \mathsf{T} \to \mathsf{Ab}$ is homological, and the functor $\mathrm{Hom}_{\mathsf{T}}(-,A)\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ is cohomological.*

Proof. The second claim follows from the first by duality so we prove only the first. So suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle. We need to show that

$$\mathrm{Hom}_{\mathsf{T}}(A,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathsf{T}}(A,Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathsf{T}}(A,Z)$$

is exact where $f_*(\theta) = f \circ \theta$ and similarly for g_*. Since $gf = 0$, it is clear that the image of f_* is contained in the kernel of g_*. Conversely, suppose that $g \circ \theta = 0$ where $\theta\colon A \to Y$. We have a diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & 0 & \longrightarrow & \Sigma A & \xrightarrow{-1} & \Sigma A \\
{\scriptstyle\theta}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle\Sigma\theta} \\
Y & \underset{g}{\longrightarrow} & Z & \underset{h}{\longrightarrow} & \Sigma X & \underset{-\Sigma f}{\longrightarrow} & \Sigma Y
\end{array}$$

in which the left square commutes as $g \circ \theta = 0$. The bottom row is a triangle by (TR2), and the top row is a triangle by (TR0) together with (TR2). Therefore, by (TR3) there exists a map $\Psi\colon \Sigma A \to \Sigma X$ making the diagram commute. Applying Σ^{-1}, one obtains that $\theta = f \circ \Sigma^{-1}\Psi$. Therefore θ is in the image of f_*, which completes the proof. □

In order to prove that certain candidate triangles are in fact triangles, it is helpful to introduce a certain subclass of homological functors, and a subclass of candidate triangles which interacts well with these homological functors.

Definition 2.12. A homological functor $H\colon \mathsf{T} \to \mathsf{A}$ is *decent* if A satisfies (AB4*) (that is, A has products and these products are exact), and H preserves products.

The relevant subclass of candidate triangles is then the following.

Definition 2.13. A candidate triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a *pretriangle* if for every decent homological functor $H\colon \mathsf{T} \to \mathsf{A}$ the induced sequence

$$\cdots \to H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}h)} HX \xrightarrow{H(f)} HY \xrightarrow{H(g)} HZ \xrightarrow{H(h)} H(\Sigma X) \to \cdots$$

is exact.

Note that every triangle is a pretriangle (Lemma 2.10) but the converse is not true.

Lemma 2.14. *Let T be a pretriangulated category, and*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'. \end{array}$$

be a morphism of pretriangles. If u and v are isomorphisms, then so is w.

Proof. Let $H\colon \mathsf{T} \to \mathsf{A}$ be a decent homological functor. Applying H gives a commutative diagram

$$\begin{array}{ccccccccc} HX & \xrightarrow{H(f)} & HY & \xrightarrow{H(g)} & HZ & \xrightarrow{H(h)} & H(\Sigma X) & \xrightarrow{H(\Sigma f)} & H(\Sigma Y) \\ \downarrow{\scriptstyle H(u)} & & \downarrow{\scriptstyle H(v)} & & \downarrow{\scriptstyle H(w)} & & \downarrow{\scriptstyle H(\Sigma u)} & & \downarrow{\scriptstyle H(\Sigma v)} \\ HX' & \xrightarrow[H(f')]{} & HY' & \xrightarrow[H(g')]{} & HZ' & \xrightarrow[H(h')]{} & H(\Sigma X') & \xrightarrow[H(\Sigma f')]{} & H(\Sigma Y') \end{array}$$

whose rows are exact since H is decent. All of the columns except for the middle are isomorphisms, and therefore by the 5 lemma, $H(w)$ is also an isomorphism. For all $A \in \mathsf{T}$, the functor $\mathrm{Hom}_{\mathsf{T}}(A, -)\colon \mathsf{T} \to \mathsf{Ab}$ is a decent homological functor by Lemma 2.11, and therefore $\mathrm{Hom}_{\mathsf{T}}(A, w)$ is an isomorphism for all $A \in \mathsf{T}$ by the previous paragraph. As such, by the Yoneda lemma w is an isomorphism as required. ☐

Proposition 2.15 (The 5 lemma)**.** *Let T be a pretriangulated category, and*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'. \end{array}$$

be a morphism of triangles. If any two of u, v, and w are isomorphisms, then so is the third.

Proof. By applying (TR2) to rotate the triangles, it suffices to prove that w is an isomorphism when u and v are both isomorphisms. This is then an immediate corollary of Lemma 2.14. ☐

Beyond proving the 5 lemma for triangles, the 5 lemma for pretriangles provides a neat way to construct new triangles from old. Recall that by (TR2) we can always produce new triangles by rotation (up to sign), and we will now see that (co)products of triangles (when the (co)products exist termwise) also yield new triangles.

Lemma 2.16. *Let T be a pretriangulated category. Suppose that $X_i \to Y_i \to Z_i \to \Sigma X_i$ is a triangle for all i. If the products exist, then the induced diagram*

$$\prod X_i \to \prod Y_i \to \prod Z_i \to \Sigma \prod X_i$$

is a triangle.

Proof. Firstly note that the above diagram makes sense since Σ commutes with all limits as it is an equivalence of categories. We next show that this diagram is a pretriangle, so fix a decent homological functor $H\colon \mathsf{T} \to \mathsf{A}$.

For each i, there is a long exact sequence

$$\cdots \to H(\Sigma^{-1}Z_i) \to H(X_i) \to H(Y_i) \to H(Z_i) \to H(\Sigma X_i) \to \cdots$$

in A. Since products are exact in A as it is (AB4*),

$$\cdots \to \prod H(\Sigma^{-1}Z_i) \to \prod H(X_i) \to \prod H(Y_i) \to \prod H(Z_i) \to \prod H(\Sigma X_i) \to \cdots$$

is also exact. Since H commutes with products, we conclude that $\prod X_i \to \prod Y_i \to \prod Z_i \to \Sigma \prod X_i$ is a pretriangle. Let us now show that it is infact a triangle.

By (TR1), we may extend the map $\prod X_i \to \prod Y_i$ to a triangle

$$\prod X_i \to \prod Y_i \to C \to \Sigma \prod X_i.$$

So by usual projection onto factors, we have for each i, a commutative diagram

$$\begin{array}{ccccccc} \prod X_i & \longrightarrow & \prod Y_i & \longrightarrow & C & \longrightarrow & \Sigma \prod X_i \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ X_i & \longrightarrow & Y_i & \longrightarrow & Z_i & \longrightarrow & \Sigma X_i \end{array}$$

so the dashed filler exists by (TR3). By universal property of the product, the maps $C \to Z_i$ assemble to give map $C \to \prod Z_i$, thus giving a commutative diagram

$$\begin{array}{ccccccc} \prod X_i & \longrightarrow & \prod Y_i & \longrightarrow & C & \longrightarrow & \Sigma \prod X_i \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ \prod X_i & \longrightarrow & \prod Y_i & \longrightarrow & \prod Z_i & \longrightarrow & \Sigma \prod X_i \end{array}$$

Both rows are pretriangles, and therefore by Lemma 2.14 we see that $\bigoplus Z_i \to C$ is an isomorphism. Therefore $\prod X_i \to \prod Y_i \to \prod Z_i \to \Sigma \prod X_i$ is isomorphic to a triangle, and hence is itself a triangle by (TR0). □

Remark 2.17. A dual argument shows that coproducts of triangles are again triangles.

Finally we have the following result, which is a standard trick for showing that a map in a triangulated category is an isomorphism.

Proposition 2.18. *Let $\theta\colon X \to Y$ be a map in T. Then θ is an isomorphism if and only if there is a triangle $X \xrightarrow{\theta} Y \to 0 \to \Sigma X$.*

Proof. This is part of Exercise A.3. □

2.5. **Functors and subcategories.** We now turn to what the 'correct' notion of functors between, and subcategories of, triangulated categories are.

Definition 2.19. Let T and U be triangulated categories. A *triangulated functor* is an additive functor $F\colon \mathsf{T} \to \mathsf{U}$ together with a natural isomorphism $\phi\colon F\Sigma \xrightarrow{\sim} \Sigma F$ such that for any triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in T, the candidate triangle

$$FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \xrightarrow{\phi_X \circ F(h)} \Sigma FX$$

is a triangle in U.

Example 2.20. The shift functor $\Sigma\colon \mathsf{T} \to \mathsf{T}$ is a triangulated functor. The identity gives a natural isomorphism $\Sigma^2 \xrightarrow{\sim} \Sigma^2$, and

$$\Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma h} \Sigma^2 X$$

is a triangle by three applications of (TR2). It is straightforward to see that this triangle is isomorphic to the candidate triangle

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \xrightarrow{\Sigma h} \Sigma^2 X$$

and hence the latter is also a triangle.

Definition 2.21. A full additive subcategory $\mathcal{S}$ of T is a *triangulated subcategory* if it is closed under isomorphisms, shifts, and triangles.

2.6. **Example: the derived category of a ring.** Returning to our motivation then, we want a category which contains all the resolutions of modules, and in which quasi-isomorphisms are isomorphisms. Since injective resolutions and projective resolutions point in opposite directions, we consider all chain complexes (i.e., rather than just those bounded above or below 0). The derived category of a ring R is the universal category in which quasi-isomorphisms of complexes are inverted. We denote this category by $\mathsf{D}(R) := \mathsf{K}(R)[\text{quasi isos}^{-1}]$. We give a precise construction of this below.

Remark 2.22. This construction of the derived category leads to some set-theoretic discussions, namely, why are the hom sets actually sets? For the purposes of this course we ignore this, and just remark that one can give alternative constructions bypassing this issue.

We can now make the definition of the derived category. We will verify that everything is well-defined afterwards.

Definition 2.23. The objects of $\mathsf{D}(R)$ are the same as the objects of $\mathsf{K}(R)$, that is, they are the chain complexes of R-modules. The morphisms in $\mathsf{D}(R)$ are equivalence classes of rooves, defined as follows. Let $M, N \in \mathsf{D}(R)$. A *roof* from M to N is a pair of chain maps

$$\begin{array}{ccc} & Z & \\ {\scriptstyle s}\swarrow{\scriptstyle \simeq} & & \searrow{\scriptstyle f} \\ M & & N \end{array}$$

where $Z \in \mathsf{D}(R)$ and s is a quasi-isomorphism. Two rooves $(M \leftarrow Z \to N)$ and $(M \leftarrow Z' \to N)$ are equivalent if there exists another roof $(M \leftarrow W \to N)$ and maps $W \to Z$ and $W \to Z'$ such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \uparrow & \searrow & \\ M & \longleftarrow & W & \longrightarrow & N \\ & \nwarrow & \downarrow & \nearrow & \\ & & Z' & & \end{array}$$

commutes. The hom sets of $\mathsf{D}(R)$ are rooves up to this equivalence relation. Composition of rooves is defined below after Lemma 2.25.

Throughout this section we use the following trivial observation: if $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a triangle in $\mathsf{K}(R)$, then f is a quasi-isomorphism if and only if Z is acyclic (i.e., $H_*(Z) = 0$). To see this, recall that f is a quasi-isomorphism if and only if the mapping cone $C(f)$ is acyclic. Since any triangle in $\mathsf{K}(R)$ is isomorphic to one in which the third term is the mapping cone, the observation follows.

Lemma 2.24 (Cancellation)**.** *For maps $f, g\colon X \to Y$ in $\mathsf{K}(R)$, the following are equivalent:*

(1) $sf = sg$ for some quasi-isomorphism s with source Y;
(2) $ft = gt$ for some quasi-isomorphism t with codomain X.

Proof. Given a quasi-isomorphism $s\colon Y \to Y'$ with $sf = sg$, we have a triangle $Z \xrightarrow{k} Y \xrightarrow{s} Y' \to \Sigma Z$ by (TR1) (and (TR2)). The functor $\operatorname{Hom}_{\mathsf{K}(R)}(X, -)$ is a homological functor by Lemma 2.11, so

$$\operatorname{Hom}_{\mathsf{K}(R)}(X, Z) \xrightarrow{k_*} \operatorname{Hom}_{\mathsf{K}(R)}(X, Y) \xrightarrow{s_*} \operatorname{Hom}_{\mathsf{K}(R)}(X, Y)$$

is exact. Since $s(f - g) = 0$, that is, $f - g \in \ker(s_*)$, there is a map $h\colon X \to Z$ such that $f - g = kh$. Consider the triangle $X' \xrightarrow{t} X \xrightarrow{h} Z \to \Sigma X'$ which exists by (TR1). As $ht = 0$, we have $(f - g)t = kht = 0$, and hence $ft = gt$ as required. So it remains to see that t is a quasi-isomorphism. Since s is a quasi-isomorphism, Z is acyclic by its defining triangle, and hence t is also a quasi-isomorphism. The other direction is analogous. □

Lemma 2.25 (The Ore Condition)**.** *Given a quasi-isomorphism $s\colon Y' \to Y$ and a map $f\colon X \to Y$ in $\mathsf{K}(R)$, there exists a commutative diagram in $\mathsf{K}(R)$*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle s'}\downarrow\simeq & & \simeq\downarrow{\scriptstyle s} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in which s' is also a quasi-isomorphism.

Proof. By (TR1) there is a triangle $Y' \xrightarrow{s} Y \xrightarrow{u} Z \to \Sigma Y'$. By (TR1) together with (TR2) there is also a triangle $X' \xrightarrow{s'} X \xrightarrow{uf} Z \to \Sigma X'$. By (TR3), there is a map $f'\colon X' \to Y'$ such that the diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{s'} & X & \xrightarrow{uf} & Z & \longrightarrow & \Sigma X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Sigma f'} \\ Y' & \xrightarrow[s]{} & Y & \xrightarrow{u} & Z & \longrightarrow & \Sigma X' \end{array}$$

commutes. Since s is a quasi-isomorphism, Z is acyclic, and hence s' is also a quasi-isomorphism. □

Lemma 2.26. *The relation on rooves defined above is an equivalence relation.*

Proof. Reflexivity and symmetry are clear, so it suffices to prove transitivity. Suppose that we have rooves $R_1 \sim R_2$ and $R_2 \sim R_3$. Spelling this out, we have two commutative diagrams

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle s}\swarrow & \uparrow{\scriptstyle p} & \searrow{\scriptstyle f} & \\ L & & W & & M \\ & {\scriptstyle t}\nwarrow & \downarrow{\scriptstyle q} & \nearrow{\scriptstyle g} & \\ & & Z' & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & Z' & & \\ & {\scriptstyle t}\swarrow & \uparrow{\scriptstyle r} & \searrow{\scriptstyle g} & \\ L & & U & & M \\ & {\scriptstyle u}\nwarrow & \downarrow{\scriptstyle v} & \nearrow{\scriptstyle h} & \\ & & V & & \end{array}$$

with sp, tq, tr and uv all quasi-isomorphisms.

By the Ore Condition (Lemma 2.25), we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{a} & U \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle tr} \\ W & \xrightarrow[sp]{} & L \end{array}$$

in which z is a quasi-isomorphism.

Note that $tqz = spz = tra$, and hence by Lemma 2.24 there is a quasi-isomorphism $\theta \colon H \to R$ such that $qz\theta = ra\theta$. Now consider the diagram

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle s}\swarrow & \uparrow{\scriptstyle pz\theta} & \searrow{\scriptstyle f} & \\ L & & H & & M \\ & {\scriptstyle u}\nwarrow & \downarrow{\scriptstyle va\theta} & \nearrow{\scriptstyle h} & \\ & & V & & \end{array}$$

The left hand square commutes since

$$spz\theta = tqz\theta = tra\theta = vua\theta$$

and the right hand square commutes as

$$fpz\theta = gqz\theta = gra\theta = hva\theta.$$

Finally, note that $spz\theta$ is a quasi-isomorphism since sp, z, and θ are. Hence $R_1 \sim R_3$ as required. □

Using the previous lemma we may now define composition in $\mathsf{D}(R)$. Given two rooves $(L \leftarrow Z \xrightarrow{f} M)$ and $(M \leftarrow Z' \xrightarrow{g} N)$, their composite gf is the roof $(L \leftarrow P \to N)$ defined via the diagram

$$\begin{array}{ccccccc} & & P & \xrightarrow{f'} & Z' & \xrightarrow{g} & N \\ & & \downarrow\simeq & & \simeq\downarrow & & \\ L & \xleftarrow[\simeq]{} & Z & \xrightarrow[f]{} & M & & \end{array}$$

where the existence of the commutative diagram is provided by the Ore condition (Lemma 2.25).

By tedious diagram chasing (left to the interested reader), one may check that the composition operation defined above is well-defined, i.e., unique up to equivalence of rooves, associative, and that the roof $X \leftarrow X \rightarrow X$ with all maps the identity, is the identity map on X. This shows that $\mathsf{D}(R)$ is a category, but we want to now show that it is infact a *triangulated* category. First we need to deal with the additive structure. In order to do this, we note a convenient way to compare rooves.

Lemma 2.27. *Let $R_i = (X \leftarrow Z_i \rightarrow Y_i)$ be a finite collection of rooves. Then there exists a quasi-isomorphism $Z \rightarrow X$ such that each R_i is equivalent to the roof $(X \leftarrow Z \rightarrow Y_i)$.*

Proof. In the case when we have two rooves, the Ore condition (Lemma 2.25) says that we have a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Z_2 \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle s_2} \\ Z_1 & \xrightarrow[s_1]{} & X \end{array}$$

in which t is a quasi-isomorphism. It is straightforward to see that the roof $X \leftarrow Z \rightarrow Y_i$ where $Z \rightarrow X$ is the composite $s_1 t$ does the trick. The case with more rooves follows by induction. ☐

Proposition 2.28. *The derived category $\mathsf{D}(R)$ is additive.*

Proof. Let us just give a sketch of how to define a group operation on the set of rooves from X to Y up to equivalence. Given rooves $(X \leftarrow Z_1 \rightarrow Y)$ and $(X \leftarrow Z_2 \rightarrow Y)$ we may replace them up to equivalence by rooves $(X \leftarrow Z \xrightarrow{f_1} Y)$ and $(X \leftarrow Z \xrightarrow{f_2} Y)$ as in Lemma 2.27. The addition of these rooves is now defined by $(X \leftarrow Z \xrightarrow{f_1+f_2} Y)$. We leave to the reader that this indeed defines an additive structure. ☐

We are now almost ready to prove that $\mathsf{D}(R)$ is a triangulated category. The missing ingredient is the following functor and a couple of its properties. We define $Q\colon \mathsf{K}(R) \rightarrow \mathsf{D}(R)$ as the functor which is the identity on objects, and which takes a morphism $f\colon X \rightarrow Y$ to the roof $X = X \xrightarrow{f} Y$.

Lemma 2.29. *The functor $Q\colon \mathsf{K}(R) \rightarrow \mathsf{D}(R)$ sends quasi-isomorphisms to isomorphisms.*

Proof. Let $f\colon X \rightarrow Y$ be a quasi-isomorphism. We claim that $R = (f, \mathrm{id}_X)\colon Y \leftarrow X = X$ is an inverse to $Q(f)$. It is easy to check that $R \circ Q(f) = \mathrm{id}_X$. Now one checks that $Q(f) \circ R = (f, f)\colon Y \leftarrow X \rightarrow Y$. The diagram

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f} & \\ Y & & & & Y \\ & {\scriptstyle \mathrm{id}_Y}\nwarrow & & \nearrow{\scriptstyle \mathrm{id}_Y} & \\ & & Y & & \end{array}$$

shows that the roof $Q(f) \circ R$ is equivalent to id_Y, and hence $Q(f)$ is an isomorphism as claimed. ☐

Lemma 2.30. *Let $f, g\colon M \rightarrow N$ in $\mathsf{K}(R)$. Then $Q(f) = Q(g)$ if and only if there exists a quasi-isomorphism t with codomain M such that $ft = gt$, if and only if there exists a quasi-isomorphism s with domain N such that $sf = sg$.*

Proof. If $Q(f) = Q(g)$, then there is a roof $M \leftarrow Z \to N$ and maps $p, q\colon Z \to M$ such that the diagram

$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle \mathrm{id}}\swarrow & \uparrow{\scriptstyle p} & \searrow{\scriptstyle f} & \\ M & \xleftarrow{\sim} & Z & \longrightarrow & N \\ & {\scriptstyle \mathrm{id}}\nwarrow & \downarrow{\scriptstyle q} & \nearrow{\scriptstyle g} & \\ & & M & & \end{array}$$

commutes. Commutativity of the left square shows that $p = q$ and that p is a quasi-isomorphism. Therefore we have a quasi-isomorphism $p\colon Z \to M$ such that $fp = gp$. Therefore by the cancellation property (Lemma 2.24) there is a quasi-isomorphism s with domain M such that $sf = sg$. The converse is analogous. ☐

Theorem 2.31. *Let R be a ring. Then the derived category $\mathsf{D}(R)$ in which the distinguished triangles are the triangles which are isomorphic in $\mathsf{D}(R)$ to those of the form*

$$M \xrightarrow{f} N \xrightarrow{i} C(f) \xrightarrow{c} \Sigma M$$

for some map of chain complexes $f\colon M \to N$, is a triangulated category. In other words, the triangles are those which are isomorphic to the images of triangles under Q.

Define a shift functor on objects as in $\mathsf{K}(R)$ and on rooves by shifting each leg of the roof. Let us now go through each of the axioms in turn.

Proof of (TR0). This is immediate since (TR0) holds in $\mathsf{K}(R)$. ☐

Proof of (TR1). Let $f\colon X \to Y$ in $\mathsf{D}(R)$. Choose a presentation $(s, a)\colon X \leftarrow Z \to Y$ of f as a roof. By the Ore condition (Exercise A.5), we obtain a commutative square

$$\begin{array}{ccc} Z & \xrightarrow{a} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[b]{} & U \end{array}$$

in which t is also a quasi-isomorphism. Using (TR1) and (TR3) in $\mathsf{K}(R)$, we have a commutative diagram

$$\begin{array}{ccccccc} Z & \xrightarrow{a} & Y & \xrightarrow{i_a} & C(a) & \xrightarrow{c_a} & \Sigma Z \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \Sigma s} \\ X & \xrightarrow[b]{} & U & \xrightarrow[i_b]{} & C(b) & \xrightarrow[c_b]{} & \Sigma X \end{array}$$

in which the rows are distinguished triangles in $\mathsf{K}(R)$. Taking the long exact sequence in homology proves that u is a quasi-isomorphism since both s and t are. We now apply Q to this diagram, and may consider the diagram

$$\begin{array}{ccccccc} Z & \xrightarrow{Q(a)} & Y & \xrightarrow{Q(i_a)} & C(a) & \xrightarrow{Q(c_a)} & \Sigma Z \\ \downarrow{\scriptstyle Q(s)} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle Q(u)} & & \downarrow{\scriptstyle \Sigma Q(s)} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[Q(i_b \circ t)]{} & C(b) & \xrightarrow[Q(c_b)]{} & \Sigma X \end{array}$$

in $\mathsf{D}(R)$. The left hand square commutes since (s, a) is a presentation of f, and the other squares commute as they are the images of commuting squares under Q. Each of the vertical maps is an isomorphism since s and u are quasi-isomorphisms, and hence we have shown that f fits into a triangle which is isomorphic to a standard one, and hence is distinguished. □

Proof of (TR2). Rotation of triangles is immediate from rotation in $\mathsf{K}(R)$. □

Proof of (TR3). The existence of fillers is not impacted by isomorphic triangles, so we may assume that we are given a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(g)} & Z & \xrightarrow{Q(h)} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow[Q(f')]{} & Y' & \xrightarrow[Q(g')]{} & Z' & \xrightarrow[Q(h')]{} & \Sigma X'. \end{array}$$

in which the left square commutes, the rows are distinguished triangles, and we need to construct a dashed map making the diagram commute. We may write the vertical maps as rooves, to give the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ {\scriptstyle s}\uparrow & & {\scriptstyle t}\uparrow & & & & {\scriptstyle \Sigma u}\uparrow \\ A & & B & & & & \Sigma A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & & & \downarrow{\scriptstyle \Sigma a} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X' \end{array}$$

in $\mathsf{K}(R)$. By the Ore condition (Lemma 2.25), we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{fs} & Y \\ {\scriptstyle t'}\uparrow & & \uparrow{\scriptstyle t} \\ A' & \xrightarrow[c]{} & B \end{array}$$

in which t' is a quasi-isomorphism.

It is easy to see that the rooves $(s, a)\colon X \leftarrow A \to X'$ and $(st', at')\colon X \leftarrow A' \to X'$ are equivalent so we may replace the left hand column in the diagram and consider the new diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle st'}\uparrow & & \uparrow{\scriptstyle t} \\ A' & \xrightarrow{c} & B \\ {\scriptstyle at'}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

The top square commutes by definition of c and t', but the bottom square need not commute in $\mathsf{K}(R)$. However, let us show that it commutes after applying Q. Since Q sends quasi-isomorphisms to isomorphisms by Lemma 2.29 we have

$$Q(f') \circ Q(a) \circ Q(t') = Q(f') \circ Q(a) \circ Q(s)^{-1} \circ Q(s) \circ Q(t').$$

Since $vQ(f) = Q(f')u$ this is in turn equal to

$$Q(b) \circ Q(t)^{-1} \circ Q(f) \circ Q(s) \circ Q(t') = Q(b) \circ Q(t)^{-1} \circ Q(t) \circ Q(c) = Q(b) \circ Q(c).$$

Therefore by Lemma 2.30, there exists a quasi-isomorphism $w\colon A'' \to A'$ such that $fat'w = bcw$. We may now again replace the left hand column in the diagram and apply (TR1) in $\mathsf{K}(R)$ to obtain the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle stw'}\uparrow & & \uparrow{\scriptstyle t} & & & & {\scriptstyle \Sigma u}\uparrow \\
A'' & \xrightarrow{cw} & B & \xrightarrow{i} & C & \xrightarrow{j} & \Sigma A \\
{\scriptstyle atw'}\downarrow & & \downarrow{\scriptstyle b} & & & & \downarrow{\scriptstyle \Sigma a} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'
\end{array}$$

in which the left square is commutative in $\mathsf{K}(R)$. Therefore, by (TR3) in $\mathsf{K}(R)$, there are fillers as indicated in the diagram below:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle stw'}\uparrow & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle \alpha} & & {\scriptstyle \Sigma u}\uparrow \\
A'' & \xrightarrow{cw} & B & \xrightarrow{i} & C & \xrightarrow{j} & \Sigma A \\
{\scriptstyle atw'}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \Sigma a} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & \Sigma X'
\end{array}$$

The only thing that remains is to check α is a quasi-isomorphism, so that $(\alpha, \beta)\colon Z \leftarrow C \to Z'$ does represent a map in $\mathsf{D}(R)$. Just as in the proof of (TR2), taking the long exact sequence in homology shows that α is a quasi-isomorphism since stw and t are both quasi-isomorphisms. □

Proof of (TR4). We omit this for brevity. □

Lemma 2.32. *Consider a roof $(s, f)\colon M \leftarrow Z \to N$. Prove that (s, f) is an isomorphism in $\mathsf{D}(R)$ if and only if f is a quasi-isomorphism, in which case $(s, f)^{-1} = (f, s)$.*

Proof. This is Exercise A.6. □

We may now say the precise way in which this construction inverts isomorphisms.

Theorem 2.33. *The functor $Q\colon \mathsf{K}(R) \to \mathsf{D}(R)$ is triangulated and has the property that $Q(s)$ is an isomorphism if s is a quasi-isomorphism. Moreover, given any functor $F\colon \mathsf{K}(R) \to \mathsf{T}$ such that $F(s)$ is an isomorphism if s is a quasi-isomorphism, there exists a unique functor $F'\colon \mathsf{D}(R) \to \mathsf{T}$ such that $F' \circ Q = F$. Moreover, if F is triangulated, then F' is also triangulated.*

Proof. The functor Q is triangulated by definition, and sends quasi-isomorphisms to isomorphisms by Lemma 2.29. For the universal property, we first prove uniqueness, so suppose that there exists such an F'. Since Q is the identity on objects, the value of F' on objects is determined by F. For maps, consider a roof (s, f) in $\mathsf{D}(R)$. Then

$$(s, f) = (\mathrm{id}, f) \circ (s, \mathrm{id}) = (\mathrm{id}, f) \circ (\mathrm{id}, s)^{-1} = Q(f) \circ Q(s)^{-1}$$

where we used Lemma 2.32 for the second equality. Therefore $F'(s, f) = F'(Q(f)) \circ F'(Q(s)^{-1}) = F(f) \circ F(s)^{-1}$, so the value of F' on maps is also determined by F.

We now show existence. Define $F'(M) = F(M)$, and $F'(s, f) = F(s)^{-1} \circ F(f)$ where we used the assumption that F sends quasi-isomorphisms to isomorphisms to deduce the existence of $F(s)^{-1}$. This is functorial since F is. (Technically we need to check that $F'(s, f)$ is invariant under equivalence of rooves; we omit this.)

We finally show that F' is triangulated if F is. Firstly, note that $(F\Sigma)' = F'\Sigma$: by uniqueness it suffices to verify that $F'\Sigma\colon \mathsf{D}(R) \to \mathsf{T}$ sends quasi-isomorphisms to isomorphisms, and satisfies $F'\Sigma Q = F\Sigma$, both of which are clear. Similarly, $(\Sigma F)' = \Sigma F'$. Therefore, $\Sigma F' = F'\Sigma$, so that F' behaves well with the shift. So suppose that

$$L \xrightarrow{Qf} M \xrightarrow{Qg} N \xrightarrow{Qh} \Sigma L$$

is a triangle in $\mathsf{D}(R)$; recall that by definition, all triangles in $\mathsf{D}(R)$ take this form. We need to check that the image under F' is a triangle in T. However, the image under T may be identified with

$$FL \xrightarrow{Ff} FM \xrightarrow{Fg} FN \xrightarrow{Fh} F\Sigma L = \Sigma FL$$

since $F'Q = F$. This is a triangle in T since F is assumed to be triangulated, and hence F' is also triangulated. □

Remark 2.34. Everything above can be vastly generalized to the case of a triangulated category T and a set S of morphisms in T which is closed under composition, and satisfies cancellation and the Ore condition. In other words, after we had verified that $\mathsf{T} = \mathsf{K}(R)$ and $S = \{\text{quasi-isomorphisms}\}$ satisfied cancellation and the Ore condition, we never used that we were working with complexes again.

The previous universal property makes precise the statement that the derived category is the universal home for homological algebra: any operation on complexes which inverts quasi-isomorphisms factors uniquely through the derived category. One can also combine this with Proposition 2.8 to construct functors on derived categories. We will use this later on when we discuss tensor-triangulated categories.

Lemma 2.35. *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules, then there is a map $h\colon N \to \Sigma L$ such that $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L$ is a triangle in $\mathsf{D}(R)$, where L, M and N are viewed as complexes in degree 0.*

Proof. The complex $C(f) = (L \xrightarrow{-f} M)$ in degrees 1 and 0 by definition. The diagram

$$\begin{array}{ccc} L & \xrightarrow{-f} & M \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & N \end{array}$$

commutes by exactness, and hence the vertical maps define a chain map $\phi\colon C(f) \to N$. This chain map is a quasi-isomorphism: $H_1(C(f)) = \ker(-f) = 0$ by exactness, and again by exactness, we have $H_0(C(f)) = M/\ker(g)$ which is isomorphic by g to N. Hence ϕ is a quasi-isomorphism.

Therefore we may define a map $h\colon N \to \Sigma L$ in $\mathsf{D}(R)$ by $c \circ \phi^{-1}$ where $c\colon C(f) \to \Sigma L$ is the canonical map. The diagram

$$\begin{array}{ccccccc} L & \xrightarrow{f} & M & \xrightarrow{i} & C(f) & \xrightarrow{c} & \Sigma L \\ \| & & \| & & \downarrow{\scriptstyle \phi} & & \| \\ L & \xrightarrow[f]{} & M & \xrightarrow[g]{} & N & \xrightarrow[h]{} & \Sigma L \end{array}$$

commutes by construction, and hence the bottom row is a triangle in $\mathsf{D}(R)$. ☐

Remark 2.36. The same argument may be generalised to prove that short exact sequences of complexes (rather than just of modules) give triangles in the derived category.

2.7. **Calculating maps in the derived category.** We end with some fundamental results about the maps in the derived category. Before we do so we introduce truncation functors; these allow inductive arguments by gluing together complexes one piece at a time. There are two forms of truncations: the so-called smart and brutal truncations.

Definition 2.37 (Brutal truncation)**.** For a complex $M \in \mathsf{D}(R)$ and integer n, the brutal truncation above n is $(t_{\geq n}M)_i = M_i$ if $i \geq n$ and 0 otherwise. Similarly, we define $(t_{\leq n}M)_i = M_i$ if $i \leq n$ and 0 otherwise. There is a short exact sequence $0 \to t_{\leq n}M \to M \to t_{\geq n+1}M \to 0$ of complexes, and hence a triangle $t_{\leq n}M \to M \to t_{\geq n+1}M \to \Sigma t_{\leq n}M$ in $\mathsf{D}(R)$. Note that this does not behave well with respect to homology, for example, the canonical map $t_{\leq n}M \to M$ does not induce an isomorphism in homology in degree n.

Definition 2.38 (Smart truncation)**.** To rectify the poor behaviour of the brutal truncation with respect to homology, it is convenient to consider an alternative form of truncation called the smart truncation. Fix a complex M and an integer M. We define complexes $\tau_{\leq n}M$ and $\tau_{\geq n}M$ as follows:

$$(\tau_{\geq n}M)_i = \begin{cases} M_i & i \geq n+1 \\ \ker(d_n\colon M_n \to M_{n-1}) & i = n \\ 0 & \text{otherwise} \end{cases}$$

$$(\tau_{\leq n}M)_i = \begin{cases} M_i & i \leq n-1 \\ \operatorname{coker}(d_{n+1}\colon M_{n+1} \to M_n) & i = n \\ 0 & \text{otherwise} \end{cases}$$

There are canonical maps $\tau_{\geq n}M \to M$ and $M \to \tau_{\leq n}M$ which induce isomorphisms on homology in degrees $\geq n$ and $\leq n$ respectively. There is a triangle

$$\tau_{\geq n}M \to M \to \tau_{\leq n-1}M \to \Sigma\tau_{\geq n}M$$

in $\mathsf{D}(R)$. This last claim needs justification. Consider the complex Q defined by

$$Q_i = \begin{cases} M_n/\ker(d_n) & i = n \\ M_i & i \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

The obvious map $\phi\colon Q \to \tau_{\leq n-1}M$ is a quasi-isomorphism. Then there is an evident short exact sequence of complexes $0 \to \tau_{\geq n}M \to M \to Q \to 0$ which gives rise to a triangle

$$\tau_{\geq n}M \to M \to Q \xrightarrow{\theta} \Sigma\tau_{\geq n}M$$

in $\mathsf{D}(R)$ by Lemma 2.35. We may then define a map $\tau_{\leq n-1}M \to \Sigma\tau_{\geq n}M$ by $\theta\phi^{-1}$. The diagram

$$\begin{array}{ccccccc} \tau_{\geq n}M & \longrightarrow & M & \longrightarrow & Q & \xrightarrow{\theta} & \Sigma\tau_{\geq n}M \\ \| & & \| & & \downarrow{\scriptstyle\phi} & & \| \\ \tau_{\geq n}M & \longrightarrow & M & \longrightarrow & \tau_{\leq n-1}M & \xrightarrow[\theta\phi^{-1}]{} & \Sigma\tau_{\geq n}M \end{array}$$

commutes, and the top row is triangle; hence the bottom row is also a triangle as required. Note that $\tau_{\geq n}\tau_{\leq n}M \simeq H_nM[n]$, so that a special case of the above triangle is $H_nM[n] \to \tau_{\leq n}M \to \tau_{\leq n-1}M$.

With truncations introduced, we may proceed with calculating maps in the derived category.

Lemma 2.39. *Let I be a bounded above complex of injective R-modules, and X be a bounded above complex. If X is acyclic, then any map $f\colon X \to I$ is null homotopic.*

Proof. Left as an exercise (Exercise A.9). □

Lemma 2.40. *Let I be a bounded above complex of injective R-modules, and X be a bounded above complex. If $s\colon I \to X$ is a quasi-isomorphism, then there is a chain map $t\colon X \to I$ such that $ts = \mathrm{id}$ in $\mathsf{K}(R)$.*

Proof. Consider the triangle $I \xrightarrow{s} X \xrightarrow{i} C(s) \xrightarrow{c} \Sigma I$. Since s is a quasi-isomorphism, the mapping cone $C(s)$ is acyclic, and hence by Lemma 2.39 the map $c\colon C(s) \to \Sigma I$ is null homotopic. Since c is null homotopic we have maps $h_n\colon C(s)_n \to (\Sigma I)_{n+1}$, which we may write as matrices $\begin{pmatrix} a_n & b_n \end{pmatrix}$ where $a_n\colon I_{n-1} \to I_n$ and $b_n\colon X_n \to I_n$. Since c is null, we have $c = d_{\Sigma I}h + hd_{C(s)}$. Writing this out in matrices, we have

$$\begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} -da & -db \end{pmatrix} + \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} -d & 0 \\ s & d \end{pmatrix} = \begin{pmatrix} bs - da - ad & bd - db \end{pmatrix}.$$

The second component says that $bd = db$ so that $b\colon X \to I$ is a chain map. The first component says that $bs - \mathrm{id} = da + ad$ and hence bs is chain homotopic to the identity as required. □

Proposition 2.41. *The localisation functor $Q\colon \mathsf{K}(R) \to \mathsf{D}(R)$ induces an isomorphism on hom groups*

$$\mathrm{Hom}_{\mathsf{K}(R)}(M, I) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}(R)}(M, I)$$

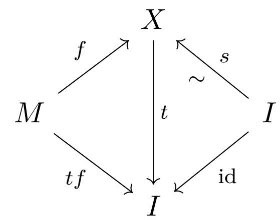
for all $M \in \mathsf{K}(R)$ whenever I is a bounded above complex of injective R-modules.

Proof. For injectivity, if $Q(f) = Q(g)$ then there is a quasi-isomorphism $s\colon I \to X$ such that $sf = sg$ by Lemma 2.30. However, by Lemma 2.40, there exists a map $t\colon X \to I$ such that $ts = \mathrm{id}_X$ in $\mathsf{K}(R)$. (Here note that we may assume that X is bounded above; it has to be bounded above in homology as I is, so write $\sup(X)$ for the highest integer for which the homology is non-zero. Then the canonical map $X \to \tau_{\leq \sup(X)}X$ is a quasi-isomorphism, so we may replace X with $\tau_{\leq \sup(X)}$.) Therefore $f = tsf = tsg = g$ so the map is injective.

For surjectivity, take any roof (here we use Exercise A.7)

$$\begin{array}{ccc} & X & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle s}\,\sim \\ M & & I \end{array}$$

By Lemma 2.40 there exists a quasi-isomorphism $t\colon X \to I$ such that $ts = \mathrm{id}$ (as before we may assume that X is bounded above). Therefore the diagram



shows that our starting roof is equivalent to $Q(tf)$. Hence the map is surjective as required. □

Lemma 2.42. *Let M be an R-module and I be a bounded above complex of injectives. Then*

$$\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i M, I) = \frac{\ker\left(\mathrm{Hom}_R(M, I_i) \xrightarrow{(d_i)_*} \mathrm{Hom}_R(M, I_{i-1})\right)}{\mathrm{im}\left(\mathrm{Hom}_R(M, I_{i+1}) \xrightarrow{(d_{i+1})_*} \mathrm{Hom}_R(M, I_i)\right)}.$$

Proof. By Proposition 2.41 we have $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i M, I) = \mathrm{Hom}_{\mathsf{K}(R)}(\Sigma^i M, I)$. The latter of these consists of homotopy classes of chain maps

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & I_{i+1} & \xrightarrow[d_{i+1}]{} & I_i & \xrightarrow[d_i]{} & I_{i-1} & \longrightarrow & \cdots \end{array}$$

A morphism of R-modules $f\colon M \to I_i$ gives such a chain map if and only if the composite $M \xrightarrow{f} I_i \xrightarrow{d_i} I_{i-1}$ is zero. Therefore

$$\mathrm{Hom}_{\mathrm{Ch}(R)}(\Sigma^i M, I) = \ker\left(\mathrm{Hom}_R(M, I_i) \xrightarrow{(d_i)_*} \mathrm{Hom}_R(M, I_{i-1})\right).$$

Now this chain map is null homotopic if and only if there exists a map $h\colon M \to I_{i+1}$ such that $f = d_{i+1}h$. Therefore the subgroup of $\mathrm{Hom}_{\mathrm{Ch}(R)}(\Sigma^i M, I)$ consisting of null homotopic maps is the image of $(d_{i+1})_*\colon \mathrm{Hom}_R(M, I_{i+1}) \to \mathrm{Hom}_R(M, I_i)$. This completes the proof of the claim. □

Proposition 2.43. *Let M and N be R-modules viewed as complexes in degree 0. Then*

$$\mathrm{Ext}^i_R(M, N) = \mathrm{Hom}_{\mathsf{D}(R)}(M, \Sigma^i N).$$

Proof. Let I be an injective resolution of N. Then

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}(R)}(M, \Sigma^i N) &= \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^{-i} M, I) && \text{as } N \simeq I \\ &= \frac{\ker\left(\mathrm{Hom}_R(M, I_{-i}) \xrightarrow{(d_{-i})_*} \mathrm{Hom}_R(M, I_{-i-1})\right)}{\mathrm{im}\left(\mathrm{Hom}_R(M, I_{-i+1}) \xrightarrow{(d_{-i+1})_*} \mathrm{Hom}_R(M, I_{-i})\right)} && \text{by Lemma 2.42} \\ &= H_{-i}\mathrm{Hom}_R(M, I) && \text{by definition of } H_{-i} \\ &= \mathrm{Ext}^i_R(M, N) && \text{by definition of Ext} \end{aligned}$$

as required. □

Finally, note that there is a dual version of Proposition 2.41, replacing bounded above complexes of injectives in the second variable with bounded below complexes of projectives in the first variable. Using this we may obtain the following reinterpretation of homology.

Proposition 2.44. *Let R be a ring and M be a complex of R-modules. Then*

$$H_i(M) = \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, M).$$

Proof. By the dual of Proposition 2.41, we have $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, M) = \mathrm{Hom}_{\mathsf{K}(R)}(\Sigma^i R, M)$. A direct calculation of this analogous to Lemma 2.42 then yields the claim. □

3. Finiteness in triangulated categories

In this section we introduce and study various notions of what it means for an object of a triangulated category to be small or finite. These are fundamental notions, and much of the theory of triangulated categories is dependent on these ideas.

3.1. **Compact objects.**

In this section we study certain types of 'small' objects in triangulated categories. It is often the case that the whole triangulated category is generated by small objects, and much of the theory of triangulated categories relies upon such assumptions.

Definition 3.1. Let T be a triangulated category which has coproducts. An object $X \in \mathsf{T}$ is said to be *compact* if the natural map

$$\bigoplus \mathrm{Hom}_{\mathsf{T}}(X, Y_i) \to \mathrm{Hom}_{\mathsf{T}}(X, \bigoplus Y_i)$$

is an equivalence for every set of objects $\{Y_i\}$. We write T^{c} (or T^{ω}) for the full subcategory of T consisting of the compact objects.

Before we can give some examples of compact objects, and criteria for detecting them, we must introduce some terminology.

Definition 3.2. Let T be a triangulated category which has coproducts.

(1) A full subcategory $\mathcal{S}$ of T is *thick* if it is closed under retracts and is triangulated.
(2) A full subcategory $\mathcal{S}$ of T is *localizing* if it is thick, and closed under coproducts.

Given a set of objects $\mathcal{X}$ of T, we write $\mathrm{Thick}(\mathcal{X})$ (resp., $\mathrm{Loc}(\mathcal{X})$) for the smallest thick (resp., localizing) subcategory of T containing $\mathcal{X}$. Note that these are well defined since the intersection of thick/localizing subcategories is again thick/localizing.

Example 3.3. Let $X \in \mathsf{T}$. The full subcategory $\{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i X, Y) \simeq 0 \text{ for all } i \in \mathbb{Z}\}$ is thick. It is localizing if X is compact.

Definition 3.4. Let $\mathcal{X}$ be a set of objects of T. We say that $\mathcal{X}$ *generates* T if $\mathrm{Loc}(\mathcal{X}) = \mathsf{T}$. If each element of $\mathcal{X}$ is compact, then we say that $\mathcal{X}$ *compactly generates* T.

If T is compactly generated, then we can characterise the compact objects in terms of building operations.

Proposition 3.5. *Let T be a triangulated category which is compactly generated by a set G. Then $\mathsf{T}^{\mathrm{c}} = \mathrm{Thick}(\mathsf{G})$.*

Proof. The implication that $X \in \mathsf{T}^{\mathrm{c}}$ implies that $X \in \mathrm{Thick}(\mathsf{G})$ requires some work so we omit it. For the converse, consider the set

$$\{Y \in \mathsf{T} \mid \bigoplus \mathrm{Hom}_{\mathsf{T}}(Y, Z_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(Y, \bigoplus Z_i) \text{ for all sets } \{Z_i\}\}.$$

This is a thick subcategory of T, and contains G. Hence it contains $\mathrm{Thick}(\mathsf{G})$ since this is the *smallest* thick subcategory of T containing G. □

Example 3.6. For a ring R, the compact objects in $\mathsf{D}(R)$ are the *perfect complexes*; that is, those complexes which are quasi-isomorphic to a bounded complex of finitely generated projectives. We will give a proof of this below in Example 3.14.

3.2. **Brown representability and consequences.** Brown Representability is a remarkably powerful result in triangulated categories. The initial statement was in stable homotopy theory where it concerns cohomology theories being represented by objects called spectra, but since then various versions have been proved in general triangulated categories. It has many striking consequences, such as providing criteria to determine existence of adjoints, a way to check compact generation, and a way to construct 'designer' objects. We will not give the proof of Brown Representability here, but we will prove a special case as Theorem 3.21.

Recall that a functor $H \colon \mathsf{T} \to \mathsf{Ab}$ is homological if it sends triangles to long exact sequences.

Example 3.7. Let X be an object of T. Then $\mathrm{Hom}_{\mathsf{T}}(X, -)$ is homological and $\mathrm{Hom}_{\mathsf{T}}(-, X)$ is cohomological by Lemma 2.11.

Theorem 3.8 (Brown Representability)**.** *Let T be a compactly generated triangulated category. If $H \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ is a cohomological functor which takes coproducts in T to products in Ab, then H is representable, i.e., it is isomorphic to $\mathrm{Hom}_{\mathsf{T}}(-, X)$ for some $X \in \mathsf{T}$.*

Recall that compactly generated triangulated categories are assumed to have coproducts by definition. The first consequence of Brown representability which we give is that in fact they also have products.

Proposition 3.9. *If T is a compactly generated triangulated category, then T has products.*

Proof. Let $\{X_i\}$ be a set of objects in T and consider the functor $H \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ defined by $H = \prod_i \mathrm{Hom}_{\mathsf{T}}(-, X_i)$. This is cohomological and sends coproducts in T to products. Hence there exists an object $Z \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{T}}(-, Z) = \prod_i \mathrm{Hom}_{\mathsf{T}}(-, X_i)$. It is an exercise to verify that Z has the universal property of the product of the X_i. □

In light of the previous proposition, we may now state a dual version of Brown representability.

Theorem 3.10 (Brown Representability for the dual)**.** *Let T be a compactly generated triangulated category. If $H \colon \mathsf{T} \to \mathsf{Ab}$ is a homological functor which takes products in T to products in Ab, then H is corepresentable, i.e., it is isomorphic to $\mathrm{Hom}_{\mathsf{T}}(Y, -)$ for some $Y \in \mathsf{T}$.*

Remark 3.11. It is important to note that despite the appearance and the terminology, the dual form of Brown representability is *not* a formal consequence of the former. Indeed, the opposite of a compactly generated triangulated category need not be compactly generated, and moreover, the existence of compact objects in the opposite category is extremely rare.

The next consequence is a powerful adjoint functor theorem.

Theorem 3.12. *Let $F\colon \mathsf{T} \to \mathsf{U}$ be a triangulated functor, and suppose that T is compactly generated.*

(1) If F is coproduct preserving then F has a right adjoint.
(2) If F is product preserving then F has a left adjoint.

Proof. We prove (1); the proof of (2) is similar. Since F is triangulated and coproduct preserving, the functor $\mathrm{Hom}_{\mathsf{U}}(F(-), Y)$ is cohomological and takes coproducts to products. Hence by Theorem 3.8, there exists an object $G(Y) \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{U}}(F(-), Y) = \mathrm{Hom}_{\mathsf{T}}(-, G(Y))$. Given a map $Y \to Y'$, one obtains a map $G(Y) \to G(Y')$ via the Yoneda lemma, namely, we have

$$\mathrm{Hom}_{\mathsf{T}}(-, G(Y)) = \mathrm{Hom}_{\mathsf{U}}(F(-), Y) \to \mathrm{Hom}_{\mathsf{U}}(F(-), Y') = \mathrm{Hom}_{\mathsf{T}}(-, G(Y'))$$

which by the Yoneda lemma must come from a map $G(Y) \to G(Y')$. One checks that this makes G into a functor which is right adjoint to F. □

Finally we give a consequence which allows one to check when a given set of compact objects is in fact a set of generators.

Proposition 3.13. *Let T be a triangulated category with coproducts, and $\mathcal{S}$ be a set of compact objects of T. Then the following are equivalent:*

(1) $\mathcal{S}$ generates T;
(2) if $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, X) = 0$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$, then $X \simeq 0$.

Proof. For the implication (1) ⇒ (2), suppose that $\mathcal{S}$ generates T, and that $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, X) = 0$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$. Consider the set

$$\mathcal{X} = \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i Y, X) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

This is a localizing subcategory of T, and contains $\mathcal{S}$ by assumption. Hence $\mathsf{T} = \mathrm{Loc}(\mathcal{S}) \subseteq \mathcal{X}$, i.e., $\mathrm{Hom}_{\mathsf{T}}(Z, X) \simeq 0$ for all $Z \in \mathsf{T}$, and so by Yoneda we have $X \simeq 0$.

For the converse, consider the localizing subcategory $\mathrm{Loc}(\mathcal{S})$ of T. Since $\mathcal{S}$ consists of compact objects, the inclusion $i\colon \mathrm{Loc}(\mathcal{S}) \hookrightarrow \mathsf{T}$ has a right adjoint Γ by Theorem 3.12. The counit of the adjunction gives a natural map $\Gamma X \to X$ which we may complete to a triangle

$$\Gamma X \to X \to Y$$

by axiom (TR1). We will show that $Y \simeq 0$, so that $\Gamma X \simeq X$ by Proposition 2.18, and hence $X \in \mathrm{Loc}(\mathcal{S})$. If we apply $\mathrm{Hom}_{\mathsf{T}}(S, -)$ to the triangle $\Gamma X \to X \to Y$ we get a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathsf{T}}(S, \Gamma X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(S, X) \to \mathrm{Hom}_{\mathsf{T}}(S, Y) \to \mathrm{Hom}_{\mathsf{T}}(\Sigma^{-1} S, \Gamma X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(\Sigma^{-1} S, X) \to \cdots$$

where the isomorphisms hold by the adjunction $i \dashv \Gamma$. Therefore $\mathrm{Hom}_{\mathsf{T}}(S, Y) \simeq 0$, and by the same argument shifted, we have $\mathrm{Hom}_{\mathsf{T}}(\Sigma^i S, Y) \simeq 0$ for all i. Therefore $Y \simeq 0$ by assumption as required. □

Example 3.14 ($\mathsf{D}(R)^{\mathrm{c}} = \mathrm{Perf}(R)$)**.** Let us show that the compact objects in $\mathsf{D}(R)$ are exactly the perfect complexes. To do this, recall from Proposition 3.5 that $\mathsf{D}(R)^{\mathrm{c}} = \mathrm{Thick}(R)$. Recall that $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^i R, -) = H_i(-)$ by Proposition 2.44. From this one checks that R is compact, and moreover R can be seen to be a generator using this together with Proposition 3.13. Since $\mathrm{Perf}(R)$ is thick (Exercise A.15) and contains R, we must have $\mathrm{Thick}(R) \subseteq \mathrm{Perf}(R)$ as $\mathrm{Thick}(R)$ is the *smallest* thick subcategory containing R. So it suffices to show that any perfect complex

P is in Thick(R). Since Thick(R) is closed under isomorphisms by definition, we may assume that P be a bounded complex of finitely generated projectives

$$P = (\cdots \to 0 \to P_a \to P_{a-1} \to \cdots \to P_b \to 0 \to \cdots).$$

We argue by induction on $a-b$. When $a-b=0$, the complex P is infact a finitely generated projective module concentrated in a single degree. Since P is finitely generated we have a surjection $f\colon R^n \to P$ and therefore a short exact sequence $0 \to \ker(f) \to R^n \to P \to 0$. Since any short exact sequence ending in a projective splits, we see that P is a summand of R^n and hence is in Thick(R). Suppose that the claim is true for $a-b=k-1$, and now fix $a-b=k$. There is a triangle

$$P_b[b] \to t_{\geq b}P \to t_{\geq b+1}P$$

by taking the brutal truncation, see Definition 2.37. By inductive hypothesis $t_{\geq b+1}P$ is in Thick(R), and we have already seen that $P_b[b]$ is also in Thick(R). Hence P is in Thick(R), which completes the proof.

3.3. **Homotopy colimits.** In this section we introduce some technical results which give constructions of objects in terms of smaller building blocks. In an ordinary category, one thinks of colimits as a way of constructing new objects from smaller pieces. However, in general triangulated categories do not admit true colimits, so the universal properties one encounters in this setting are weaker than the usual universal properties of colimits.

Before we give the key definition, let us give some motivation. Suppose we have a system of abelian groups and group homomorphisms $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$. The map

$$\bigoplus_{i=0}^{j-1} A_i \xrightarrow{1-f} \bigoplus_{i=0}^{j} A_i$$

which on the nth component is given by

$$A_n \xrightarrow{a \mapsto (a, -f_n(a))} A_n \oplus A_{n+1} \xrightarrow{\text{inc}} \bigoplus_{i=0}^{j} A_i$$

gives rise to an exact sequence

$$0 \to \bigoplus_{i=0}^{j-1} A_i \xrightarrow{1-f} \bigoplus_{i=0}^{j} A_i \to A_j \to 0.$$

Taking the direct limit of this (and since direct limits are exact in abelian groups), we obtain an exact sequence

$$0 \to \bigoplus A_i \xrightarrow{1-f} \bigoplus A_i \to \varinjlim A_i \to 0. \tag{3.15}$$

We may mimic this exact sequence in a triangulated category to define the homotopy colimit.

Definition 3.16. Let T be a triangulated category with coproducts, and let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system of maps in T. The *homotopy colimit* of this system, denoted $\text{hocolim} X_i$, is the cone of the map

$$\bigoplus X_i \xrightarrow{1-f} \bigoplus X_i$$

which on the nth component is given by

$$X_n \xrightarrow{x \mapsto (x, -f_n(x))} X_n \oplus X_{n+1} \xrightarrow{\text{inc}} \bigoplus X_i.$$

Lemma 3.17. *Let T be a triangulated category with coproducts, and let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system of maps in T. Let $Y \in \mathsf{T}$, and suppose that there exists maps $g_n\colon X_n \to Y$ such that $g_n = g_{n+1}f_n$ for all n. Then there exists a map $\overline{g}\colon \mathrm{hocolim} X_i \to Y$ such that $g = \overline{g}q$ where q is the induced map $q\colon \bigoplus X_i \to \mathrm{hocolim} X_i$.*

Proof. By universal property of the coproduct, we obtain a map $g\colon \bigoplus X_i \to Y$ by assembling the g_i. Note that $g(1-f) = 0$ since on the nth component we have $g(1-f)(x) = g(x_n, -f_n(x_n)) = g_n(x_n) - g_{n+1}f_n(x_n) = 0$. So consider the diagram

$$\begin{array}{ccccccc} \bigoplus X_i & \xrightarrow{1-f} & \bigoplus X_i & \xrightarrow{q} & \mathrm{hocolim} X_i & \longrightarrow & \Sigma \bigoplus X_i \\ \downarrow & & \downarrow g & & & & \downarrow \\ 0 & \longrightarrow & Y & \xrightarrow[\mathrm{id}]{} & Y & \longrightarrow & 0. \end{array}$$

The top row is a triangle by definition of the homotopy colimit, and the bottom row is a triangle by (TR0). The left hand square commutes since $g(1-f) = 0$, so by (TR3) there exists a map $\overline{g}\colon \mathrm{hocolim} X_i \to Y$ making the diagram commute. □

Lemma 3.18. *Let T be a triangulated category with coproducts. Let C be a compact object and let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system in T. Then the canonical map*

$$\varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i)$$

is an isomorphism.

Proof. By definition of the homotopy colimit and since $\mathrm{Hom}(C, -)$ is a homological functor by Lemma 2.11 we obtain a long exact sequence,

$$\cdots \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus \Sigma X_i) \to \cdots.$$

By shifting the exact sequence (3.15), and comparing with the long exact sequence above, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) & \longrightarrow & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) & \longrightarrow & \varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) & \longrightarrow & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) & \longrightarrow & \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) & \longrightarrow & \cdots \end{array}$$

The first two columns are isomorphisms since C is compact, and so we see that the map $\mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i)$ is injective. Similarly, we may apply the same argument to ΣX_i to deduce that $\mathrm{Hom}_{\mathsf{T}}(C, \bigoplus \Sigma X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus \Sigma X_i)$ is also injective. Therefore the long exact sequence collapses into short exact sequences. We then have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) & \longrightarrow & \bigoplus \mathrm{Hom}_{\mathsf{T}}(C, X_i) & \longrightarrow & \varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) & \longrightarrow & \mathrm{Hom}_{\mathsf{T}}(C, \bigoplus X_i) & \longrightarrow & \mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) & \longrightarrow & 0 \end{array}$$

in which the first two columns are isomorphisms as C is compact. The result then follows. □

Corollary 3.19. *Let T be a compactly generated triangulated category. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots$ be a system in T, and let $Y \in \mathsf{T}$ such that there exists maps $g_n\colon X_n \to Y$ with*

$g_n = g_{n+1} f_n$ for all n. If the canonical map

$$\varinjlim \mathrm{Hom}_{\mathsf{T}}(C, X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, Y)$$

is an isomorphism for all $C \in \mathsf{T}^{\mathrm{c}}$, then $Y \simeq \mathrm{hocolim} X_i$.

Proof. By Lemma 3.17 we obtain a map $\overline{g}\colon \mathrm{hocolim} X_i \to Y$. Since T is compactly generated, to prove this is an isomorphism it suffices to check that the induced map

$$\mathrm{Hom}_{\mathsf{T}}(C, \mathrm{hocolim} X_i) \to \mathrm{Hom}_{\mathsf{T}}(C, Y)$$

is an isomorphism for all $C \in \mathsf{T}^{\mathrm{c}}$. This is immediate from Lemma 3.18. □

Before we can prove some consequences of this definition, we need the following construction. The following construction is the key behind the proof of the Brown representability theorem. Although we won't use it to prove the full version of Brown representability stated in the previous section, Theorem 3.21 can be interpreted as a special case.

Construction 3.20. Let T be a triangulated category with coproducts, and suppose that $\mathcal{K} = \{K_i\}$ is a set of compact objects in T. Let $X \in \mathsf{T}$ be arbitrary. We will build a tower whose homotopy colimit defines a colocalization functor, and hence a localization functor (we haven't introduced this language yet, but we will recast this construction in that terminology later on in the course). Let $X_0 = X$ and

$$A_0 = \bigoplus_{Z \in \Sigma^? \mathcal{K}} \bigoplus_{Z \to X} Z$$

where the question mark indicates that this ranges across all possible shifts. There is a map $A_0 \to X_0$ by universal property of the coproduct, and we write X_1 for the cofibre of this map. Iterating the same procedure, we set

$$A_i = \bigoplus_{Z \in \Sigma^? \mathcal{K}} \bigoplus_{Z \to X_i} Z$$

and a triangle $A_i \to X_i \to X_{i+1}$. Write F_i for the fibre of the map $X \to X_i$.

Therefore we have triangles $F_i \to X \to X_i$ and $A_i \to X_i \to X_{i+1}$ for each i. Note that the map $X \to X_{i+1}$ factors as $X \to X_i \to X_{i+1}$. Therefore by applying the octahedral axiom to the triangles $X \to X_i \to \Sigma F_i$, $X_i \to X_{i+1} \to \Sigma A_i$, and $X \to X_{i+1} \to \Sigma F_{i+1}$ we obtain a triangle $\Sigma F_i \to \Sigma F_{i+1} \to \Sigma A_i$. Pictorially, this is

$$\begin{array}{ccccccc}
X & \longrightarrow & X_i & \longrightarrow & \Sigma F_i & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle 1} & & \downarrow & & \vdots & & \downarrow{\scriptstyle 1} \\
X & \longrightarrow & X_{i+1} & \longrightarrow & \Sigma F_{i+1} & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow{\scriptstyle 1} & & \vdots & & \downarrow \\
X_i & \longrightarrow & X_{i+1} & \longrightarrow & \Sigma A_i & \longrightarrow & \Sigma X_i \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\
\Sigma F_i & \dashrightarrow & \Sigma F_{i+1} & \dashrightarrow & \Sigma A_i & \dashrightarrow & \Sigma^2 F_i
\end{array}$$

Shifting we therefore have a system of maps $\cdots \to F_i \to F_{i+1} \to \cdots$. We define $\Gamma_{\mathcal{K}} X$ to be the homotopy colimit of the system of F_i's. Note that by applying Lemma 3.17, we have a map $\Gamma_{\mathcal{K}} X \to X$.

Theorem 3.21. *With notation as in Construction 3.20, we have $\Gamma_{\mathcal{K}} X \in \mathsf{Loc}(\mathcal{K})$, and the induced map $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma_{\mathcal{K}} X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is an isomorphism for all $Z \in \mathsf{Loc}(\mathcal{K})$. In particular, $\Gamma_{\mathcal{K}} X \to X$ is an isomorphism if $X \in \mathsf{Loc}(\mathcal{K})$.*

Proof. For the first claim it suffices to show that each $F_i \in \mathsf{Loc}(\mathcal{K})$ by definition of the homotopy colimit. We do this by induction. The base case is clear as $F_0 \simeq 0$, so suppose $F_i \in \mathsf{Loc}(\mathcal{K})$. By construction, there is a triangle $F_i \to F_{i+1} \to A_i$ and A_i is a coproduct of suspensions of elements of $\mathcal{K}$. Therefore $F_{i+1} \in \mathsf{Loc}(\mathcal{K})$ as required.

For the second claim, by a localizing subcategory argument it suffices to prove it when $Z \in \mathcal{K}$.

Let us first deal with surjectivity. For any $i > 0$, the map $\mathrm{Hom}_{\mathsf{T}}(Z, X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X_i)$ is zero by construction: any such Z embeds into A_{i-1} and hence by the triangle $A_{i-1} \to X_{i-1} \to X_i$ the map is zero. Therefore the map $\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is surjective by the exact sequence associated to the triangle $F_i \to X \to X_i$ by Lemma 2.11. From the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{T}}(Z, F_i) & \twoheadrightarrow & \mathrm{Hom}_{\mathsf{T}}(Z, X) \\ \downarrow & \nearrow & \\ \mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) & & \end{array}$$

it follows that $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is surjective. (Recall that if gf is surjective, then g is surjective.)

To complete the proof of the second claim, we need to verify that the map is injective. So suppose that $f\colon Z \to \Gamma X$ is such that $Z \to \Gamma X \to X$ is zero. Now $f \in \mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) = \varinjlim \mathrm{Hom}_{\mathsf{T}}(Z, F_i)$ by Lemma 3.18, and so there exists an $f_i\colon Z \to F_i$ such that f factors as

$$Z \xrightarrow{f_i} F_i \to \Gamma X.$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(Z, F_i) & & \\ {\scriptstyle\alpha}\downarrow & \searrow & \\ \mathrm{Hom}(Z, \Gamma X) & \xrightarrow[\beta]{} & \mathrm{Hom}(Z, X). \end{array}$$

By definition $\alpha(f_i) = f$, and by construction $\beta(f) = 0$. Therefore f_i is in the kernel of the map $\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$. By exactness of the sequence associated to the triangle $F_i \to X \to X_i$,

$$\ker(\mathrm{Hom}_{\mathsf{T}}(Z, F_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)) = \mathrm{im}(\mathrm{Hom}_{\mathsf{T}}(\Sigma^{-1} Z, X_i) \to \mathrm{Hom}_{\mathsf{T}}(Z, F_i)).$$

Therefore there exists a map $h\colon Z \to \Sigma^{-1} X_i$ such that the composite $Z \to \Sigma^{-1} X_i \to F_i$ is f_i. By definition of A_i and the triangle

$$\Sigma^{-1} A_i \xrightarrow{\delta} F_i \xrightarrow{g} F_{i+1},$$

the map h factors through δ and hence $g f_i = 0$. Finally write $\bar{g}$ for the map $F_{i+1} \to \Gamma X$. We then have

$$f = (\bar{g} \circ g) \circ f_i = \bar{g} \circ (g \circ f_i) = 0$$

so that the map is injective as required. This completes the proof of the second claim.

For the final claim, if $X \in \mathsf{Loc}(\mathcal{K})$, then as $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma_{\mathcal{K}}X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is an isomorphism for all $Z \in \mathsf{Loc}(\mathcal{K})$ we see that $\Gamma_{\mathcal{K}}X \to X$ is an isomorphism by the Yoneda lemma. ☐

Remark 3.22. An extended version of the argument used in the proof of the previous theorem can be used to give a proof of Brown representability (Theorem 3.8).

We obtain the following interesting consequence of this construction which gives a characterisation of the localizing subcategory generated by a set of compact objects.

Corollary 3.23. *Let T be a triangulated category with coproducts. Suppose that $\mathcal{K}$ is a set of compact objects in T. Then the following are equivalent for an arbitrary object $X \in \mathsf{T}$:*

(1) X is the homotopy colimit of a sequence

$$0 = F_0 \to F_1 \to F_2 \to \cdots$$

such that the cofibre of $F_i \to F_{i+1}$ for each i is a coproduct of suspensions of elements of $\mathcal{K}$;

(2) $X \in \mathsf{Loc}(\mathcal{K})$.

Proof. The implication (1) ⇒ (2) holds similarly to Theorem 3.21. For (2) ⇒ (1), we use Construction 3.20. By Theorem 3.21, $\Gamma_{\mathcal{K}}X \to X$ is an isomorphism as $X \in \mathsf{Loc}(\mathcal{K})$. As $\Gamma_{\mathcal{K}}X$ is a homotopy colimit of the required form, this completes the proof. ☐

We will use this construction extensively later on, since it provides a recipe for constructing objects with convenient properties from small building blocks.

4. Tensor-triangulated categories and finiteness

Triangulated categories abound, but often there is extra structure floating around which it is useful to remember; we consider when the triangulated category is also equipped with a closed symmetric monoidal structure which is suitably compatible with the triangulation. Such considerations have exploded into the field of *tensor-triangular geometry* whose starting point was the observation that a scheme $\mathfrak{X}$ cannot be recovered from the structure of $\mathsf{D}(\mathrm{QCoh}(\mathfrak{X}))$ as a triangulated category alone, but it can be recovered once the tensor product is remembered.

Recall that a symmetric monoidal structure on a category $\mathcal{C}$ is the data of a bifunctor

$$- \otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

called the *tensor product* (or *monoidal product*) together with an unit object $\mathbb{1} \in \mathcal{C}$ satisfying:

- unitality: $\mathbb{1} \otimes X \simeq X$ for all $X \in \mathcal{C}$;
- associativity: $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ for all $X, Y, Z \in \mathcal{C}$;
- symmetry: $X \otimes Y \simeq Y \otimes X$ for all $X, Y \in \mathcal{C}$;

all satisfying various coherences that we won't make precise here. Such a monoidal structure is moreover called *closed*, if there is a bifunctor $\underline{\mathrm{Hom}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ called the *internal hom* such that $- \otimes X$ is left adjoint to $\underline{\mathrm{Hom}}(X, -)$. Again, this is all subject to various coherences that we will not make explicit. Note that $\underline{\mathrm{Hom}}(\mathbb{1}, X) \simeq X$ for all $X \in \mathcal{C}$.

Definition 4.1. A *tensor-triangulated category* T is a triangulated category which also has a closed symmetric monoidal structure, such that the tensor product and internal hom are triangulated functors in both variables.

4.1. **Example: the derived category.** Let R be a *commutative* ring. Recall the tensor product of chain complexes of R-modules is given by

$$(M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j$$

with differential $d(m \otimes n) = (dm \otimes n) + (-1)^{|m|}(m \otimes dn)$. However, this cannot be a tensor product on the derived category since it is not invariant under quasi-isomorphism as the following example shows.

Example 4.2. Let $R = \mathbb{Z}/4$ and define $P = (\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \cdots)$. There is a quasi-isomorphism $P \xrightarrow{\sim} 0$. However, this is not preserved by the functor $\mathbb{Z}/2 \otimes_{\mathbb{Z}/4} -$. Indeed $\mathbb{Z}/2 \otimes_{\mathbb{Z}/4} P$ is the complex

$$\cdots \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \cdots$$

and hence has homology $\mathbb{Z}/2$ in each degree.

So we now explain how to give a construction of a tensor product which is invariant under quasi-isomorphism, thus making the derived category into a tensor-triangulated category.

In a similar vein, it is important to note that if P is a complex of projective R-modules, it does not preserve quasi-isomorphisms. For example, consider $R = \mathbb{Z}/4$ and the complex

$$P = \cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \cdots.$$

The quasi-isomorphism $P \xrightarrow{\sim} 0$ is not preserved by $\mathrm{Hom}_{\mathbb{Z}/4}(P, -)$. As such, we need to introduce refined notions of projective and injective objects in order to construct functors between derived categories. So as to not get bogged down in homological algebra which is unrelated to the triangulated structure, we will omit many of the proofs of the following claims.

Definition 4.3. A complex P of R-modules is *dg-projective* if any of the following equivalent conditions hold:

(i) $\mathrm{Hom}_R(P, -)$ preserves surjective quasi-isomorphisms;
(ii) $\mathrm{Hom}_R(P, -)$ is exact and preserves quasi-isomorphisms;
(iii) P is a complex of projective R-modules and $\mathrm{Hom}_R(P, -)$ preserves acyclic complexes.
(iv) for any chain map $f\colon P \to N$ and any surjective quasi-isomorphism $g\colon M \to N$, there exists a chain map $\alpha\colon P \to M$ so that $g\alpha = f$.

The following gives the appropriate notion of a projective resolution in this context.

Theorem 4.4. *For any complex of R-modules M, there exists a dg-projective complex P together with a surjective quasi-isomorphism $P \xrightarrow{\sim} M$.*

In a similar way, we have the following for injectives instead.

Definition 4.5. A complex I of R-modules is *dg-injective* if any of the following equivalent conditions hold:

(i) $\mathrm{Hom}_R(-, I)$ sends injective quasi-isomorphisms to surjective quasi-isomorphisms;
(ii) $\mathrm{Hom}_R(-, I)$ is exact and preserves quasi-isomorphisms;
(iii) I is a complex of injective R-modules and $\mathrm{Hom}_R(-, I)$ preserves acyclic complexes.
(iv) for any chain map $f\colon M \to I$ and any injective quasi-isomorphism $g\colon M \to N$, there exists a chain map $\alpha\colon N \to I$ so that $\alpha g = f$.

Theorem 4.6. *For any complex of R-modules M, there exists a dg-injective complex I together with an injective quasi-isomorphism $M \xrightarrow{\sim} I$.*

There is also the case for flat objects.

Definition 4.7. A complex F of R-modules is *dg-flat* if any of the following equivalent conditions hold:

(i) $F \otimes_R -$ sends injective quasi-isomorphisms to surjective quasi-isomorphisms;
(ii) $F \otimes_R -$ is exact and preserves quasi-isomorphisms;
(iii) F is a complex of flat R-modules and $F \otimes_R -$ preserves acyclic complexes.

Theorem 4.8. *For any complex of R-modules M, there exists a dg-flat complex F together with a quasi-isomorphism $F \xrightarrow{\sim} M$.*

Construction 4.9. Using the above, we can construct dg-projective and dg-injective resolutions functors $P\colon \mathsf{K}(R) \to \mathsf{K}(R)$ and $I\colon \mathsf{K}(R) \to \mathsf{K}(R)$ as follows. We explain the version for dg-projective resolutions; the dg-injective case is analogous. Given any $M \in \mathsf{K}(R)$, set $P(M)$ to be the dg-projective complex with surjective quasi-isomorphism $P(M) \to M$ as given by Theorem 4.4. We note that this is unique up to homotopy equivalence: by Definition 4.3(iv), any two dg-projective resolutions of M are quasi-isomorphic, and just as in the classical case, quasi-isomorphisms between dg-projectives are in fact homotopy equivalences. Given a map $f\colon M \to N$ in $\mathsf{K}(R)$, we define $P(f)$ to be the map given by Definition 4.3(iv) applied to the diagram

$$\begin{array}{ccc} P(M) & \longrightarrow & M \\ {\scriptstyle P(f)}\downarrow & & \downarrow{\scriptstyle f} \\ P(N) & \longrightarrow & N. \end{array}$$

Note that P sends quasi-isomorphisms to isomorphisms since quasi-isomorphisms between dg-projectives are homotopy equivalences. One can moreover prove that this is a triangulated functor.

With all these preliminaries set up, we can now discuss derived tensor products and derived homs. These are in fact special case of something more general: *(total) derived functors.*

Proposition 4.10. *Let $F\colon \mathsf{K}(R) \to \mathsf{K}(S)$ be a functor. There exists functors $\mathsf{L}F\colon \mathsf{D}(R) \to \mathsf{D}(S)$ and $\mathsf{R}F\colon \mathsf{D}(R) \to \mathsf{D}(S)$ called the total left (resp. right) derived functors of F. If F is triangulated, then so are $\mathsf{L}F$ and $\mathsf{R}F$.*

Proof. Consider the functor $Q \circ F \circ P\colon \mathsf{K}(R) \to \mathsf{D}(S)$. This sends quasi-isomorphisms to isomorphisms since P does, and hence by Theorem 2.33 there exists a unique functor $\mathsf{L}F\colon \mathsf{D}(R) \to \mathsf{D}(S)$ so that $\mathsf{L}F \circ Q = Q \circ F \circ P$. Moreover, if F is triangulated, then so is the composite $Q \circ F \circ P$ and hence so is $\mathsf{L}F$. The existence of $\mathsf{R}G$ is analogous by instead considering the functor $Q \circ F \circ I\colon \mathsf{K}(R) \to \mathsf{D}(S)$. ☐

Henceforth we assume that R is a commutative ring. One can make some of the following statements for non-commutative rings if one worries about left vs right module structures, but we will focus only on the commutative case.

Construction 4.11. Let M be a complex. The functor $M \otimes - \colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is homotopy invariant so by Proposition 2.8 there exists a functor $M \otimes - \colon \mathsf{K}(R) \to \mathsf{K}(R)$. The total left derived functor of this (in the sense of Proposition 4.10) is the derived tensor product functor $M \otimes^{\mathsf{L}}_R - \colon \mathsf{D}(R) \to \mathsf{D}(R)$.

The following result is fundamental and we will use it throughout. Recall that the tensor product of complexes is given by $(M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j$.

Proposition 4.12. *The derived tensor product $M \otimes^{\mathsf{L}}_R -$ constructed in Construction 4.11 may be computed using a dg-flat replacement F of M, i.e., $M \otimes^{\mathsf{L}}_R - = F \otimes_R -$, and is independent of the choice of such a replacement.*

Proof. Suppose that F is a dg-flat replacement of M, and write $G \colon \mathsf{D}(R) \to \mathsf{D}(R)$ for the induced functor making the diagram

$$\begin{array}{ccc} \mathsf{K}(R) & \xrightarrow{F \otimes_R -} & \mathsf{K}(S) \\ {\scriptstyle Q_R}\downarrow & & \downarrow{\scriptstyle Q_S} \\ \mathsf{D}(R) & \xrightarrow[G]{} & \mathsf{D}(S) \end{array}$$

commute. Since $M \otimes^{\mathsf{L}}_R -$ satisfies a uniqueness property, it suffices to check that the diagram

$$\begin{array}{ccc} \mathsf{K}(R) & \xrightarrow{P(M) \otimes_R -} & \mathsf{K}(S) \\ {\scriptstyle Q_R}\downarrow & & \downarrow{\scriptstyle Q_S} \\ \mathsf{D}(R) & \xrightarrow[G]{} & \mathsf{D}(S) \end{array}$$

commutes.

By Definition 4.3(iv), there exists a quasi-isomorphism $P(M) \xrightarrow{\phi} F$. For any $X \in \mathrm{Ch}(R)$, we claim that $\phi \otimes_R X \colon P(M) \otimes_R X \to F \otimes_R X$ is a quasi-isomorphism. Indeed, take a dg-flat replacement $\psi \colon F(X) \to X$ and consider the commutative diagram

$$\begin{array}{ccc} P(M) \otimes_R F(X) & \xrightarrow{P(M) \otimes_R \psi} & P(M) \otimes_R X \\ {\scriptstyle \phi \otimes_R F(X)}\downarrow & & \downarrow{\scriptstyle \phi \otimes_R X} \\ F \otimes_R F(X) & \xrightarrow{F \otimes_R \psi} & F \otimes_R X. \end{array}$$

The left vertical and both horizontals are quasi-isomorphisms since tensoring with a semi-flat complex preserves quasi-isomorphisms. Therefore the right most vertical is also a quasi-isomorphism as claimed. Since Q sends quasi-isomorphisms to isomorphisms, we see that $Q_S(P(M) \otimes_R -) = Q_S(F \otimes_R -)$, and hence by uniqueness, $G = M \otimes^{\mathsf{L}}_R -$. ☐

In a similar way we construct the derived hom functor. Recall that the internal hom functor for complexes is given by $\mathrm{Hom}_R(M, N)_n = \prod_{i \in \mathbb{Z}} (M_i, N_{i+n})$.

Construction 4.13. Let M be a complex. The functor $\mathrm{Hom}_R(M, -) \colon \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is homotopy invariant, and therefore by Proposition 4.10, there exists a triangulated functor $\mathsf{R}\mathrm{Hom}_R(M, -) \colon \mathsf{D}(R) \to \mathsf{D}(R)$, the total right derived functor of $\mathrm{Hom}_R(M, -)$. Note therefore that $\mathsf{R}\mathrm{Hom}_R(M, -)(N) = \mathrm{Hom}_R(M, I(N))$.

We can also do this on the other side.

Construction 4.14. Let N be a complex. Associated to $\mathrm{Hom}_R(-,N)$ is a triangulated functor $\mathsf{R}\mathrm{Hom}_R(-,N)\colon \mathsf{D}(R)^{\mathrm{op}} \to \mathsf{D}(R)$; more precisely, as in Proposition 4.10 but now with op's, we consider the functor $Q \circ \mathrm{Hom}_R(-,N) \circ P^{\mathrm{op}}\colon \mathsf{K}(R)^{\mathrm{op}} \to \mathsf{D}(R)$. Note that $\mathsf{R}\mathrm{Hom}_R(-,N)(M) = \mathrm{Hom}_R(P(M),N)$.

Proposition 4.15. *Let $M, N \in \mathsf{D}(R)$. Let P be a dg-projective replacement of M, and I be a dg-injective replacement of N. Then $\mathrm{Hom}_R(P,N)$ and $\mathrm{Hom}_R(M,I)$ are quasi-isomorphic. As such, there is no ambiguity in the definition of $\mathsf{R}\mathrm{Hom}_R(-,-)$: it can be computed by resolving either factor.*

Proof. We have quasi-isomorphisms $\mathrm{Hom}_R(P,N) \to \mathrm{Hom}_R(P,I)$ and $\mathrm{Hom}_R(M,I) \to \mathrm{Hom}_R(P,I)$ by definition of dg-projective and dg-injective. □

With all these preliminaries we may prove that $\mathsf{D}(R)$ (for R commutative) is a tensor-triangulated category.

Proposition 4.16. *Let R be a commutative ring. Then $\mathsf{D}(R)$ is a tensor-triangulated category.*

Proof. We skip over most of the technical coherence details one should check in the definition of a symmetric monoidal category. We have a functor $- \otimes_R^{\mathsf{L}} -$ by Construction 4.11 which equips $\mathsf{D}(R)$ with a symmetric monoidal structure. We see that $M \otimes_R^{\mathsf{L}} -$ is left adjoint to $\mathsf{R}\mathrm{Hom}_R(M,-)$, since we may take a semi-projective replacement P of M, and compute both using this. It is standard that $P \otimes_R -$ is left adjoint to $\mathrm{Hom}_R(P,-)$ at the level of chain complexes, and the claim follows from this. Finally both the derived tensor and derived hom functors are triangulated in both variables by construction. □

4.2. **Rigid objects.** When T does not (necessarily) have a tensor product, we saw that compactness is a measure of smallness. When T is tensor-triangulated, there are various other notions of smallness in play. Under reasonable assumptions these notions are closely related as we will see below.

Recall that the counit of the tensor-hom adjunction is the evaluation map $\mathrm{ev}\colon X \otimes F(X,Y) \to Y$ and the unit is the coevaluation map $\mathrm{coev}\colon Y \to F(X, X \otimes Y)$.

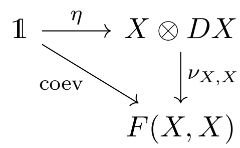
Definition 4.17. Suppose that T is a tensor-triangulated category with coproducts.

(1) An object $X \in \mathsf{T}$ is said to be *rigid* if the natural map
$$\nu_{X,Y}\colon F(X,\mathbb{1}) \otimes Y \to F(X,Y)$$
is an equivalence for all $Y \in \mathsf{T}$. To spell it out, the natural map is the composite
$$F(X,\mathbb{1}) \otimes Y \xrightarrow{\mathrm{coev}} F(X, X \otimes F(X,\mathbb{1}) \otimes Y) \xrightarrow{F(X,\mathrm{ev})} F(X,Y).$$

(2) An object $X \in \mathsf{T}$ is said to be *F-compact* if the natural map
$$\bigoplus F(X,Y_i) \to F(X, \bigoplus Y_i)$$
is an equivalence for every set $\{Y_i\}$.

In order to study the properties of rigid objects it is helpful to consider an alternative definition (which we will show is in fact equivalent).

Definition 4.18. Let T be a tensor-triangulated category, and $X \in \mathsf{T}$. The object X is *dualizable* if and only if there exists a map $\eta\colon \mathbb{1} \to X \otimes DX$ such that the diagram



commutes.

We write $D\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ for the functor $D = F(-,\mathbb{1})$. This is called the *functional dual* (or sometimes Spanier-Whitehead dual), as justified by the following results which shows that it gives an equivalence on the full subcategory of dualizable objects.

Firstly, note that there is a natural map $\rho\colon X \to D^2X$ given by the composite

$$X \xrightarrow{\mathrm{coev}} F(DX, X \otimes DX) \xrightarrow{F(DX,\mathrm{ev})} F(DX,\mathbb{1})$$

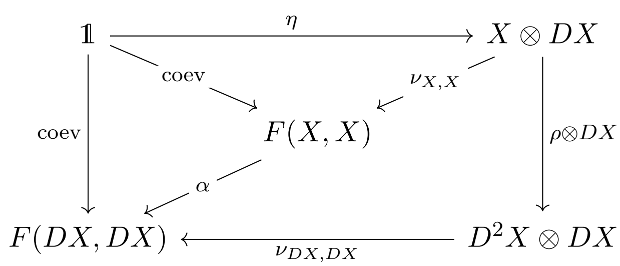
for all $X \in \mathsf{T}$. Many of the proofs of the following statements follow by tedious diagram chases, so we provide only an outline of the proofs.

Lemma 4.19. *Let T be a tensor-triangulated category and $X \in \mathsf{T}$. If X is dualizable then DX is dualizable.*

Proof. We define a map $\mathbb{1} \to DX \otimes D^2X$ via the composite

$$\mathbb{1} \xrightarrow{\eta} X \otimes DX \xrightarrow{\rho \otimes DX} D^2X \otimes DX.$$

It remains to check that the required diagram commutes. That is, we want to check that the outer square in the diagram



commutes, where α is defined to be the composite

$$F(X,X) \xrightarrow{F(X,\rho)} F(X,D^2X) \simeq F(X \otimes DX,\mathbb{1}) \simeq F(DX,DX).$$

The top triangle commutes since X is dualizable, so it suffices to prove that the other subdiagrams commute. This can be checked by unravelling the definitions of α and ρ; we leave this as a (tedious) exercise for the reader. $\square$

Lemma 4.20. *Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is dualizable. Then the natural map $\rho\colon X \to D^2X$ is an equivalence.*

Proof. We claim that the composite

$$D^2X \simeq \mathbb{1} \otimes D^2X \xrightarrow{\eta \otimes D^2X} X \otimes DX \otimes D^2X \xrightarrow{X \otimes \mathrm{ev}} X$$

is an inverse to ρ. This can be checked by diagram chasing. $\square$

Proposition 4.21. *Let T be a tensor-triangulated category, and $X \in \mathsf{T}$. Then X is rigid if and only if it is dualizable.*

Proof. The forward direction is clear since we may take $\eta = \nu_{X,X}^{-1} \circ \mathrm{coev}$. For the reverse direction, we claim that the composite

$$F(X,Y) \simeq F(X,Y) \otimes \mathbb{1} \xrightarrow{F(X,Y)\otimes\eta} F(X,Y) \otimes X \otimes DX \xrightarrow{\mathrm{ev}\otimes DX} Y \otimes DX$$

is an inverse to $\nu_{X,Y}$. Again, one may verify this by diagram chasing. □

Using Lemma 4.19 and Lemma 4.20, we therefore see that rigid objects are closed under D, and that D is a duality on rigid objects. One other important property of rigid objects is the following.

Lemma 4.22. *Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is rigid. Then X is a retract of $X \otimes DX \otimes X$.*

Proof. Recall that $X \otimes -$ is left adjoint to $F(X,-) \simeq DX \otimes -$ as X is rigid. By the triangle identity (on the unit $\mathbb{1}$), the diagram

$$\begin{array}{ccc} X & \xrightarrow{X\otimes\eta} & X \otimes DX \otimes X \\ & \searrow^{=} & \downarrow \varepsilon \\ & & X \end{array}$$

commutes, which is exactly the claim that X is a retract of $X \otimes DX \otimes X$. □

Is it natural to ask how the different notions of 'smallness' in Definition 3.1 and Definition 4.17 are related. The following answers this.

Proposition 4.23. *Let T be a tensor-triangulated category, and suppose that T has a set of rigid generators G.*

(1) For $X \in \mathsf{T}$ we have
$$X \in \mathrm{Thick}(\mathsf{G}) \implies X \text{ is rigid} \iff X \text{ is } F\text{-compact.}$$
(2) If the elements of G are compact, then for $X \in \mathsf{T}$ we have
$$X \text{ is compact} \iff X \in \mathrm{Thick}(\mathsf{G}) \implies X \text{ is rigid} \iff X \text{ is } F\text{-compact.}$$
(3) If the elements of G are compact, and the unit $\mathbb{1}$ of T is compact, then for $X \in \mathsf{T}$ we have
$$X \text{ is compact} \iff X \in \mathrm{Thick}(\mathsf{G}) \iff X \text{ is rigid} \iff X \text{ is } F\text{-compact.}$$

Proof. For (1), firstly note that the set of rigid objects in T is thick (Exercise A.19). By assumption it contains G, so we have $\mathrm{Thick}(\mathsf{G}) \subseteq \{\text{rigid objects}\}$, which proves the first implication. That rigid objects are F-compact is an immediate consequence of the definitions. For the remaining implication, suppose X is F-compact, and consider the set

$$\mathcal{L} = \{Y \in \mathsf{T} \mid DX \otimes Y \xrightarrow{\sim} F(X,Y)\}.$$

The set $\mathcal{L}$ is localizing as X is F-compact, and contains G by Exercise A.20. Therefore $\mathcal{L} = \mathsf{T}$, and hence X is rigid.

For (2), it suffices to prove that X is compact if and only if $X \in \mathrm{Thick}(\mathsf{G})$, which was the content of Proposition 3.5. We leave (3) as an exercise. □

Definition 4.24. When T satisfies the assumptions of Proposition 4.23(3) (that is, T has a set of rigid and compact generators and the unit is compact), we say that T is a rigidly-compactly generated tensor-triangulated category.

Example 4.25. Let R be a commutative ring. Then the derived category $\mathsf{D}(R)$ is a rigidly-compactly generated tensor-triangulated category. To see this, note that $\mathsf{D}(R)$ is generated by R by Proposition 2.44 and Proposition 3.13, and that R is both compact and rigid. So this follows from Proposition 4.23(3). More generally, suppose that T is a tensor-triangulated category which is compactly generated by the tensor unit $\mathbb{1}$. Then T is a rigidly-compactly generated tensor-triangulated category.

Finally, it is convenient to introduce terminology for the appropriate analogues of thick and localizing subcategories in the tensor-triangular setting.

Definition 4.26. Let T be a tensor-triangulated category. A thick/localizing subcategory $\mathcal{S}$ is a $\otimes$-*ideal* if for all $X \in \mathcal{S}$ and $Y \in \mathsf{T}$ we have $X \otimes Y \in \mathcal{S}$. We write $\mathrm{Thick}^{\otimes}(-)$ and $\mathrm{Loc}^{\otimes}(-)$ for the smallest thick and localizing $\otimes$-ideal generated by a set of objects.

5. Local cohomology in algebra

5.1. **Matlis duality.** We will state Matlis duality which is a fundamental result in commutative algebra. In order to state Matlis duality, we need to recall some definitions and facts regarding injective modules, and completions. We will simply state the key facts, and not give proofs since the course is dedicated to triangulated categories rather than commutative algebra. One may find proofs in most of the standard algebra textbooks.

Fix a commutative ring R. Recall that an R-module E is *injective* if $\mathrm{Hom}_R(-, E)$ is an exact functor, or equivalently, if for any injective map $f\colon M \hookrightarrow N$ of R-modules and any map $g\colon M \to E$ there exists a map $h\colon N \to E$ so that $g = hf$.

For any R-module M, there exists an R-module $E(M)$ satisfying the following properties:

(1) $E(M)$ is an injective R-module;
(2) $M \subseteq E(M)$ and for any non-zero submodule N of $E(M)$, we have $N \cap M \neq 0$.

Equivalently, $E(M)$ is the minimal injective containing M, so that if $M \subseteq I \subseteq E(M)$ for some injective I, then $I = E(M)$. Such an $E(M)$ is called the *injective hull* of M. Note that the injective hull also depends on the ring, so sometimes it is necessary to write $E_R(-)$ for clarity.

Let I be an ideal in R, and let M be an R-module. There are inclusions $I^n \subseteq I^{n-1}$ for all n, and therefore surjections $p_n\colon M/I^nM \to M/I^{n-1}M$. Taking the limit of this system (in the category of R-modules) defines the I-*adic completion* of the module M:

$$M_I^{\wedge} := \lim_n M/I^nM.$$

In other words, an element of $M_I^{\wedge}$ is a sequence (m_n) with $m_n \in M/I^nM$ such that $p_n(m_n) = m_{n-1}$. Here it is important to warn the reader that I-adic completion has poor homological behaviour, for example, it is not an exact functor. However, it is exact when restricted to finitely generated modules by the Artin-Rees lemma.

Recall that limits in the category of rings are created by the forgetful functor, and therefore $R_I^\wedge$ is itself a ring. It comes equipped with a canonical map $R \to R_I^\wedge$, and if this map is an isomorphism we say that R is *I-adically complete*.

Definition 5.1. Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring. Define a functor $(-)^\vee \colon \mathsf{Mod}(R)^{\mathrm{op}} \to \mathsf{Mod}(R)$ by the assignment $M^\vee = \mathrm{Hom}_R(M, E(k))$. This is called the *Matlis dual*.

Theorem 5.2 (Matlis duality)**.** *The assignment $(-)^\vee$ yields a bijection from Artinian R-modules to finitely generated $R_\mathfrak{m}^\wedge$-modules with inverse $N \mapsto \mathrm{Hom}_{R_\mathfrak{m}^\wedge}(N, E(k))$.*

5.2. **Derived functors.** We will give a brief recap on the left and right derived functors of additive functors. We give the definitions in terms of general abelian categories, but we will only need the case of module categories, so the reader may focus on that case if they wish.

Definition 5.3. Let A and B be abelian categories, and suppose that A has enough projectives. Given an additive functor $F \colon \mathsf{A} \to \mathsf{B}$, its left derived functors $L_iF \colon \mathsf{A} \to \mathsf{B}$ are defined by

$$L_iF(M) = H_i(F(P))$$

where P is a projective resolution of M. Dually, if A has enough injectives, its right derived functors R^iF are defined by $R^iF(M) = H_{-i}(F(I))$ where I is an injective resolution of M.

Remark 5.4. Note that our homological grading convention forces the minus sign in the definition of R^iF since the injective resolution is concentrated in non-positive degrees.

Example 5.5. For a fixed R-module M, we have the right exact functor $M \otimes_R -$ whose left derived functors are $\mathrm{Tor}_i^R(M \otimes -)$. Similarly, we have the left exact functor $\mathrm{Hom}_R(M, -)$ whose right derived functors are $\mathrm{Ext}_R^i(M, -)$. Equivalently, $\mathrm{Ext}_R^i(M, N)$ may be computed as $L_i(\mathrm{Hom}_R(-, N))(M)$.

Left and right derived functors satisfy the following properties:

Proposition 5.6. *Let $F \colon \mathsf{A} \to \mathsf{B}$ be an additive functor, and suppose that A has enough projectives or injectives accordingly.*

(1) If F is left exact, then $R^0F = F$. If F is right exact, then $L_0F = F$.
(2) Given a short exact sequence $0 \to L \to M \to N \to 0$, there are long exact sequences

$$0 \to (R^0F)(L) \to (R^0F)(M) \to (R^0F)(N) \to (R^1F)(L) \to (R^1F)(M) \to \cdots$$

and

$$\cdots \to (L_1F)(M) \to (L_1F)(N) \to (L_0F)(L) \to (L_0F)(M) \to (L_0F)(N) \to 0$$

in derived functors.

Left and right derived functors also satisfy a universal property which characterises them up to some properties. We only state the version for right derived functors; the version for left derived functors is dual. In order to state the universal property, we need to recall a definition.

Definition 5.7. Let A and B be abelian category. A *cohomological δ-functor* is a collection of functors $G^n \colon \mathsf{A} \to \mathsf{B}$ for each $n \geq 0$, such that for any short exact sequence $0 \to L \to M \to N \to 0$ there exists a connecting map $\delta \colon G^n(N) \to G^{n+1}(L)$ satisfying the following conditions:

(1) there is a long exact sequence

$$\cdots \to G^{n-1}(N) \to G^n(L) \to G^n(M) \to G^n(N) \to G^{n+1}(L) \to \cdots$$

(2) δ is natural, so that given a map of short exact sequences the obvious square involving δ commutes.

Note that for example that right derived functors assemble into a cohomological δ-functor. A map of cohomological δ-functors is a collection of natural transformations $\theta^n\colon G^n \to H^n$ which are compatible with the connecting maps.

We can now state the universal property of right derived functors.

Proposition 5.8. *Let A and B be abelian categories and suppose that A has enough injectives. Let $F\colon \mathsf{A} \to \mathsf{B}$ be an additive functor, and let $G\colon \mathsf{A} \to \mathsf{B}$ be a cohomological δ-functor. If there exists a natural map $\theta^0\colon R^0(F) \to G^0$, then there is a unique extension of θ^0 to a map of cohomological δ-functors $\{\theta^n\}$. Moreover, if θ^0 is an isomorphism and $G^n(E) = 0$ for all $n > 0$ and injective E, then each θ^n is an isomorphism.*

5.3. **Local cohomology.** We fix a commutative ring R and an ideal I of R. Recall that for a positive integer n, I^n denotes the ideal generated by all n-fold products of elements of I.

Definition 5.9. The *I-power torsion functor* $T_I\colon \mathsf{Mod}(R) \to \mathsf{Mod}(R)$ is defined by

$$T_I M = \{x \in M \mid I^n x = 0 \text{ for some } n > 0\}.$$

There is a canonical inclusion $T_I M \hookrightarrow M$ for each R-module M, and if this map is an isomorphism, we say that M is I-power torsion.

Note that this means that for each $i \in I$, there exists some n such that $i^n x = 0$, rather than each x being annihilated by a power of *some* i.

Let us record some key properties of the I-power torsion functor in the following lemma/exercise.

Lemma 5.10. *Let R be a commutative ring and I an ideal of R.*

(1) If $f\colon M \to N$ is a map of R-modules, then $f(T_I M) \subseteq T_I N$ and hence T_I is a functor $\mathsf{Mod}(R) \to \mathsf{Mod}(R)$.
(2) The functor T_I is left exact.
(3) In general T_I is not exact.
(4) If I and J are ideals such that $\sqrt{I} = \sqrt{J}$ then $T_I = T_J$.

Proof. This is left as Exercise A.28. ∎

Definition 5.11. Let M be an R-module. The *ith local cohomology module* of M is defined by

$$H_I^i(M) = (R^i T_I)(M)$$

where R^i denotes the ith right derived functor. More explicitly, $H_I^i(M) = H_{-i}(T_I(E))$ where E is an injective resolution of M.

It is worth noting a couple of immediate properties which are clear from the definition and the properties of right derived functors:

- The local cohomology depends on the ideal I only up to radical; that is, if $\sqrt{I} = \sqrt{J}$, then $H_I^*(-) = H_J^*(-)$.
- Since T_I is left exact, we have $H_I^0(M) = T_I M$.
- Each element of $H_I^i(M)$ is annihilated by a power of I.

- Given a short exact sequence $0 \to L \to M \to N \to 0$ there is a long exact sequence
$$0 \to T_I L \to T_I M \to T_I N \to H^1_I(L) \to H^1_I(M) \to H^1_I(N) \to H^2_I(L) \to \cdots$$
in local cohomology.

Example 5.12. Let's calculate the local cohomology groups $H^*_{(p)}(\mathbb{Z})$. An injective resolution of $\mathbb{Z}$ is given by the complex $E = (\mathbb{Q} \to \mathbb{Q}/\mathbb{Z})$. Therefore $T_{(p)}E = (0 \to T_{(p)}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/p^\infty)$. The latter follows since one may identify $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}/p^\infty$. Therefore,
$$H^0_{(p)}(\mathbb{Z}) = 0 \quad \text{and} \quad H^1_{(p)}(\mathbb{Z}) = \mathbb{Z}/p^\infty.$$

Remark 5.13. We remark that in the previous example, the local cohomology $H^*_{\mathfrak{m}}(R)$ was concentrated in the degree $\dim(R)$, and moreover, is the injective hull of the residue field $R/\mathfrak{m}$. This is not a coincidence - indeed, a commutative Noetherian local ring $(R, \mathfrak{m})$ is Gorenstein if and only if $H^*_{\mathfrak{m}}(R)$ is concentrated in degree $\dim(R)$ where it is the injective hull of the residue field. There is also a similar characterisation of Cohen-Macaulay rings: a commutative Noetherian local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if and only if $H^*_{\mathfrak{m}}(R)$ is concentrated in a single degree.

Let us now assume that R is Noetherian. Therefore the ideal I is finitely generated, so we may write $I = (x_1, \cdots, x_n)$.

Definition 5.14. The *stable Koszul complex* $K_\infty(I)$ is defined by
$$K_\infty(I) = \bigotimes_{i=1}^n (R \to R[1/x_i])$$
where the complex $R \to R[1/x]$ is in degrees 0 and -1.

Remark 5.15. We first must justify why the previous definition is independent of the choice of a set of generators. To see this, firstly note that if $\alpha \in I$, then the canonical map $K_\infty(I, \alpha) \to K_\infty(I)$ is a quasi-isomorphism (see Exercise A.30). So if y and z are sequences of elements that both generate I, then we have quasi-isomorphisms
$$K_\infty(\mathsf{y}) \xleftarrow{\sim} K_\infty(\mathsf{y}, \mathsf{z}) \xrightarrow{\sim} K_\infty(\mathsf{z})$$
as required. Moreover, since $K_\infty(x) \simeq K_\infty(x^i)$ for any i, we see that the stable Koszul complex is independent of the ideal up to radical, i.e., if $\sqrt{I} = \sqrt{J}$ then $K_\infty(I) = K_\infty(J)$.

Using the stable Koszul complex, we may give an alternative characterization of local cohomology which is often easier to use in practice.

Theorem 5.16. *Let R be a commutative Noetherian ring and I be an ideal. Let M be an R-module. There is a natural isomorphism*
$$H^i_I(M) \cong H_{-i}(K_\infty(I) \otimes_R M)$$
for all i.

Proof. Recall the universal property of right derived functors from Proposition 5.8. Since $H_{-i}(K_\infty(I) \otimes_R -)$ is a cohomological δ-functor, it suffices to prove:

(1) $H_0(K_\infty(I) \otimes_R M) = T_I M$ for all R-modules M;
(2) $H_{-i}(K_\infty(I) \otimes_R E) = 0$ for all $i > 0$ and E injective.

For the first of these, by definition

$$H_0(K_\infty(I)\otimes_R M) = H_0\left(0 \to M \xrightarrow{f} \bigoplus_{i=1}^n M[1/x_i] \to \cdots\right) = \ker(f).$$

Recall that in $S^{-1}R$, we have $q/r = x/y$ if and only if $s(qy - rx) = 0$ for some $s \in S$. An element is in the kernel of f if and only if $m/1 = 0$ in each localization. By definition, $m/1 = 0$ in $M[1/x_i]$ if and only if $x_i^j m = 0$ for some j. As such we see that $\ker(f) = T_I M$.

For the second, we will only prove the case when I is generated by a single element. (One can then argue by induction to obtain the general case, but this is quite complicated, so we omit it here.) So suppose that $I = (x)$. Then as R is Noetherian, by considering the chain of ideals

$$\ker(\cdot x) \subseteq \ker(\cdot x^2) \subseteq \ker(\cdot x^3) \subseteq \cdots$$

there exists some a such that $x^{a+1}r = 0$ implies $x^a r = 0$ for all $r \in R$. Therefore the map $R \to (x) \oplus R/(x^a)$ defined by $r \mapsto (xr, [r])$ is injective. Indeed, if $xr = 0$ and $[r] = 0$ in $R/(x^a)$, then $r \in (x^a)$ so $r = x^a s$ for some $s \in R$. Now $0 = xr = x^{a+1}s$ implies that $r = x^a s = 0$ by above as required.

Since E is an injective module, any map $f\colon R \to E$ extends to a map $(g_f, h_f)\colon (x) \oplus R/(x)^a \to E$ for which $f(1) = g_f(x) + h_f(1)$. Again since E is injective and we have an injection $(x) \to R$, the map $g_f\colon (x) \to E$ extends to a map $g'_f\colon R \to E$ such that $g_f(x) = g'_f(x) = xg'_f(1)$. Therefore $f(1) = xg'_f(1) + h_f(1)$, and note that $h_f(1)$ is (x)-torsion.

Now, any $e \in E$ defines a map $f_e\colon R \to E$ via $f_e(1) = e$. Let us write g_e for g_{f_e} and similarly for h. Therefore for any $e \in E$ we have $e = xg'_e(1) + h_e(1)$, where $g'_e(1) \in E$ and $h_e(1) \in T_{(x)}E$.

Note that we are trying to prove that $H_{-1}(E \to E[1/x]) = \operatorname{coker}(E \to E[1/x]) = 0$, or in other words, that $E \to E[1/x]$ is surjective. So we take $e/x^m \in E[1/x]$ and need to show that $e/x^m = e'/1$ for some $e' \in E$. Since $e = xg_e(1) + t$ for some torsion element t, there is a number n for which $x^n e = x^{n+1}g_e(1)$. Therefore $e/x^m = xg_e(1)/x^m$. Iterating this procedure, $e/x^m = x^m e'/x^m = e'/1$ for some $e' \in E$, so $E \to E[1/x]$ is surjective as required. ☐

From the previous theorem, one immediately sees that if I can be generated up to radical by n elements, then $H^i_I(M) = 0$ for all $i > n$ and all M, since the stable Koszul complex is concentrated in degrees $-n$ to 0.

At this point we can give the classical algebraic statement and proof of Grothendieck local duality which we will vastly generalize in the remainder of the course. Recall that a local Noetherian ring $(R, \mathfrak{m}, k)$ is *Gorenstein* if $H^{\dim(R)}_{\mathfrak{m}}(R) = E(k)$ and $H^i_{\mathfrak{m}}(R) = 0$ otherwise.

Theorem 5.17 (Grothendieck local duality)**.** *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring. Then*

$$\operatorname{Ext}^i_R(M, R^\wedge_{\mathfrak{m}}) = H^{\dim(R)-i}_{\mathfrak{m}}(M)^\vee$$

for all R-modules M.

Proof. We write $d = \dim(R)$ for brevity. By Theorem 5.16, $H^{d-i}_{\mathfrak{m}}(M) = H_{i-d}(K_\infty(\mathfrak{m}) \otimes_R M)$. Since R is Gorenstein, $\Sigma^d K_\infty(\mathfrak{m})$ is a flat resolution of $H^d_{\mathfrak{m}}(R) = E(k)$; it is quasi-isomorphic to $H^d_{\mathfrak{m}}(R)$ by Theorem 5.16, and is a complex of flat modules by definition. The shift is just to ensure that the stable Koszul complex lives in the correct degrees (between d and 0, rather than between 0 and $-d$) so that it is indeed a flat resolution. Therefore

$$H_{i-d}(K_\infty(\mathfrak{m}) \otimes_R M) = H_i(\Sigma^d K_\infty(\mathfrak{m}) \otimes_R M) = \operatorname{Tor}^R_i(M, E(k)).$$

Applying the Matlis dual and Exercise A.27, we have

$$H^{d-i}_{\mathfrak{m}}(M)^\vee = \mathrm{Tor}^R_i(M, E(k))^\vee = \mathrm{Ext}^i_R(M, E(k)^\vee) = \mathrm{Ext}^i_R(M, R^\wedge_{\mathfrak{m}})$$

as required. □

There is also the following dual form of Grothendieck local duality.

Corollary 5.18. *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring. Then*

$$\mathrm{Ext}^i_R(M, R)^\vee = H^{\dim(R)-i}_{\mathfrak{m}}(M)$$

for all finitely generated R-modules M.

Proof. This is Exercise A.32. □

Remark 5.19. Taken together, these two forms of Grothendieck local duality say that over complete local rings, $H^{\dim(R)-i}_{\mathfrak{m}}(M)$ and $\mathrm{Ext}^i_R(M, R)$ are Matlis dual for finitely generated modules.

6. A universal property for local cohomology

In this section we revisit the construction of local cohomology from a more abstract point of view, and see it as some sort of universal operation. This universality will allow us to reframe local cohomology in a purely triangulated category setting in the next section.

6.1. **Building and local cohomology.** Let R be a commutative ring, and $I = (x_1, \cdots, x_n)$ be a finitely generated ideal. One frequently encounters the quotient ring R/I when doing commutative algebra, but from the perspective of triangulated categories, this isn't well behaved since it is rarely compact. However, one may replace R/I by a compact object up to the ambient structure of the triangulated category, as we will make precise with the next result. We say that X builds Y if $Y \in \mathrm{Loc}(X)$.

Definition 6.1. Let R be a commutative ring and $I = (x_1, \cdots, x_n)$ be a finitely generated ideal. The *unstable Koszul complex* $K(I)$ is defined by

$$K(I) = \bigotimes_{i=1}^{n} (R \xrightarrow{\cdot x} R)$$

where the complex $R \xrightarrow{\cdot x} R$ is in degrees 0 and -1. Note that by definition, this is a compact object in $\mathsf{D}(R)$.

For an element $x \in R$, we note that we have a map $K(x^i) \to K(x^{i+1})$ for all i, represented by the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\cdot x^i} & R \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \cdot x} \\ R & \xrightarrow[\cdot x^{i+1}]{} & R \end{array}$$

Therefore we have a tower $\cdots \to K(x^i) \to K(x^{i+1}) \to \cdots$ and we may take the homotopy colimit of this.

Lemma 6.2. *Let $x \in R$. Then* $\mathrm{hocolim}(K(x^i)) \simeq K_\infty(x)$.

Proof. By Corollary 3.19 it suffices to show that we have compatible maps $K(x^i) \to K_\infty(x)$ for all i, so that the induced map

$$\varinjlim \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^j R, K(x^i)) \to \mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^j R, K_\infty(x))$$

is an isomorphism for all j. The existence of compatible maps is clear from the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\cdot x^i} & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R[1/x]. \end{array}$$

Recall from Proposition 2.44 that $\mathrm{Hom}_{\mathsf{D}(R)}(\Sigma^j R, -) = H_j(-)$, so it suffices to treat the cases $j = 0$ and $j = -1$. For $j = -1$, $H_{-1}(K(x^i)) = R/x^i$ and $H_{-1}(K_\infty(x)) = R[1/x]/R = R/x^\infty$. Therefore we see that the claim holds for $j = -1$. For $j = 0$, $H_0(K(x^i)) = \{m \mid x^i m = 0\}$ and $H_0(K_\infty(x)) = \{m \mid x^i m = 0 \text{ for some } i\}$. As such we see that $\varinjlim H_0(K(x^i)) = H_0(K_\infty(x))$ which completes the proof of the claim. ☐

Theorem 6.3. *Let R be a commutative ring and $I = (x_1, \cdots, x_n)$ be a finitely generated ideal. Then*

$$\mathrm{Loc}(R/I) = \mathrm{Loc}(K(I)) = \mathrm{Loc}(K_\infty(I)).$$

Proof. We will prove that (1) R/I builds $K(I)$, (2) $K(I)$ builds $K_\infty(I)$, and (3) $K_\infty(I)$ builds R/I. Note that this suffices since it implies that

$$\mathrm{Loc}(R/I) \subseteq \mathrm{Loc}(K_\infty(I)) \subseteq \mathrm{Loc}(K(I)) \subseteq \mathrm{Loc}(R/I).$$

(1) For each $i \in \mathbb{Z}$ there is a triangle

$$H_i(K(I)) \to \tau_{\leq i} K(I) \to \tau_{\leq i-1} K(I)$$

by Definition 2.38. We now claim that the homology groups of $K(I)$ are modules over R/I. To see this, note that multiplication by x is null homotopic as a map $K(x) \to K(x)$. As such, for any $x \in I$, $x\colon K(I) \to K(I)$ is null homotopic. Therefore x acts as 0 in homology, and as such the homology groups of $K(I)$ are all R/I-modules. Therefore $H_i(K(I))$ may be built from R/I in $\mathsf{D}(R/I)$. The restriction of scalars functor $\mathsf{D}(R/I) \to \mathsf{D}(R)$ is triangulated and preserves coproducts, and hence R/I also builds $H_i(K(I))$ in $\mathsf{D}(R)$. Now we argue by induction that $\tau_{\leq i} K(I)$ is built from R/I, for each $-n \leq i \leq 0$. We start with $i = -n$ as the base case. When $i = -n$, $\tau_{\leq i-1} K(I) \simeq 0$ so the base case holds by the above triangle. Then inducting, we obtain that $\tau_{\leq i} K(I)$ is built from R/I for each $-n \leq i \leq 0$. When $i = 0$, $\tau_{\leq i} K(I) = K(I)$ so this gives the claim as desired.

(2) Note that $K_\infty(x)$ is the homotopy colimit of $K(x^i)$ by Lemma 6.2. Therefore by combining Corollary 3.23 with the triangle $K(x^{i-1}) \to K(x^i) \to K(x)$ from Exercise A.39, we see that $K_\infty(x)$ lies in the localizing subcategory generated by $K(x)$. As such, $K(x)$ builds $K_\infty(x)$. The general case follows from this.

(3) One sees from Definition 4.7 that $K_\infty(I)$ is a dg-flat complex, and hence $R/I \otimes^{\mathsf{L}}_R K_\infty(I) = R/I \otimes_R K_\infty(I)$ by Proposition 4.12. As $R/I \otimes_R R[1/x_i] \simeq 0$, one sees that $R/I \otimes^{\mathsf{L}}_R K_\infty(I) \simeq R/I$. As R builds any M in $\mathsf{D}(R)$, $K_\infty(I) \simeq K_\infty(I) \otimes^{\mathsf{L}}_R R$ builds $K_\infty(I) \otimes^{\mathsf{L}}_R M$ for all $M \in \mathsf{D}(R)$. Therefore $K_\infty(I)$ builds $R/I \otimes^{\mathsf{L}}_R K_\infty(I) \simeq R/I$ as required.

Therefore we have the above chain of inclusions of localising subcategories, and hence they are all equal. □

Remark 6.4. The previous result shows that 'up to the ambient structure of $\mathsf{D}(R)$' the objects R/I, $K(I)$ and $K_\infty(I)$ are all equivalent. This is an incredibly powerful point of view which is frequently encountered. For example, one can never hope to classify all objects of $\mathsf{D}(R)$ up to quasi-isomorphism, so we instead try to classify them up to some operations (triangles, sums, products, retracts, and subsets of these). There are very structured ways to study such questions in terms of support theories which we will not touch on here. Some keywords here for those who are interested are tensor-triangular geometry, the Balmer spectrum, stratification, and costratification.

Corollary 6.5. *The following are equivalent for a complex $M \in \mathsf{D}(R)$:*

1. $K(I) \otimes^{\mathsf{L}}_R M \simeq 0$
2. $K_\infty(I) \otimes^{\mathsf{L}}_R M \simeq 0$
3. $R/I \otimes^{\mathsf{L}}_R M \simeq 0$
4. $\mathsf{R}\mathrm{Hom}_R(K(I), M) \simeq 0$
5. $\mathsf{R}\mathrm{Hom}_R(K_\infty(I), M) \simeq 0$
6. $\mathsf{R}\mathrm{Hom}_R(R/I, M) \simeq 0$

Proof. Note that $\{N \in \mathsf{D}(R) \mid N \otimes^{\mathsf{L}}_R M \simeq 0\}$ is localizing, and so the first three conditions are equivalent by Theorem 6.3. A similar argument gives that the latter three conditions are equivalent. So it suffices to see that (1) and (4) are equivalent. As $K(I)$ is compact (=rigid), $\mathsf{R}\mathrm{Hom}_R(K(I), M) \simeq DK(I) \otimes^{\mathsf{L}}_R M$. Now $DK(I) \simeq \Sigma^{\ell(I)} K(I)$ where $\ell(I)$ denotes the number of generators for I (Exercise A.41), and the claim follows. □

Theorem 6.6. *The inclusion $\mathrm{Loc}(K(I)) \hookrightarrow \mathsf{D}(R)$ admits a right adjoint Γ_I. Moreover, $\Gamma_I \simeq K_\infty(I) \otimes^{\mathsf{L}}_R -$. Hence when R is Noetherian, $H_{-i}(\Gamma_I M) = H^i_I(M)$ for all R-modules M.*

Proof. The existence of the right adjoint follows from Theorem 3.12. By uniqueness of adjoints, it suffices to verify that $K_\infty(I) \otimes^{\mathsf{L}}_R - \colon \mathsf{D}(R) \to \mathrm{Loc}(K(I))$ is right adjoint to the inclusion. Firstly, we note that this is well-defined (i.e., the image of $K_\infty(I) \otimes^{\mathsf{L}}_R -$ is in $\mathrm{Loc}(K(I))$) by Theorem 6.3 together with the fact that R is a generator for $\mathsf{D}(R)$.

There is a map $K_\infty(I) \to R$, so we want to show that the induced map

$$\mathrm{Hom}(N, K_\infty(I) \otimes^{\mathsf{L}}_R M) \xrightarrow{\sim} \mathrm{Hom}(N, M)$$

is an isomorphism for all $M \in \mathsf{D}(R)$ and all $N \in \mathrm{Loc}(K(I))$. Equivalently, that

$$f_{N,M} \colon \mathsf{R}\mathrm{Hom}_R(N, K_\infty(I) \otimes^{\mathsf{L}}_R M) \xrightarrow{\sim} \mathsf{R}\mathrm{Hom}_R(N, M)$$

is an equivalence for all $M \in \mathsf{D}(R)$ and all $N \in \mathrm{Loc}(K(I))$. The set of $N \in \mathsf{D}(R)$ such that $f_{N,M}$ is an equivalence is a localizing subcategory, so it suffices to show this for $N = K(I)$. From here, by another localising subcategory argument we may further reduce to showing it for $M = R$.

So we want to show that $\mathsf{R}\mathrm{Hom}_R(K(I), K_\infty(I)) \to \mathsf{R}\mathrm{Hom}_R(K(I), R)$ is an equivalence. Now by Corollary 6.5, this is moreover equivalent to checking that $R/I \otimes^{\mathsf{L}}_R K_\infty(I) \to R/I$ is an equivalence which we saw is true in the proof of Theorem 6.3.

The final part of the statement then follows from Theorem 5.16. □

Remark 6.7. In the previous proof, we made a passage from the external abelian-group valued hom to the internal hom. This was possible since R (the monoidal unit) is a generator for $\mathsf{D}(R)$. Without this, the argument would fail, but would be fixed if we replaced Loc by $\mathrm{Loc}^{\otimes}$ everywhere. This is a subtle but important point when it comes to generalising this theory beyond $\mathsf{D}(R)$ as we will see later on.

Definition 6.8. We call the functor $\Gamma_I\colon \mathsf{D}(R) \to \mathsf{D}(R)$ the *derived I-torsion functor*. An object $M \in \mathsf{D}(R)$ is said to be *derived torsion* if the canonical map $\Gamma_I M \to M$ is an equivalence. We denote the full subcategory of derived I-torsion objects by $\mathsf{D}(R)^{I\text{-tors}}$.

Corollary 6.9. *The derived torsion functor Γ_I is idempotent, that is, the natural map $\Gamma_I\Gamma_I M \to \Gamma_I M$ is an equivalence for all $M \in \mathsf{D}(R)$.*

Proof. There are several ways to see this. One way is to note that since $\Gamma_I\colon \mathsf{D}(R) \to \mathsf{D}(R)^{I\text{-tors}}$ is right adjoint to a fully faithful functor by Theorem 6.6, the counit map $\Gamma_I N \to N$ is an equivalence for all $N \in \mathsf{D}(R)^{I\text{-tors}}$. Such N are of the form $\Gamma_I N$ by assumption, which gives the claim.

Alternatively, one can just make a direct computation since it suffices to show that $K_\infty(I) \otimes^{\mathsf{L}}_R K_\infty(I) \to K_\infty(I)$ is an equivalence. Again, there are several ways to do this using either the arguments in Theorem 6.3 or Theorem 6.6, and we leave it to the reader to choose their favourite. □

6.2. **Derived completion and local homology.** The identification of derived torsion as a tensor product suggests the following dual definition.

Definition 6.10. The *derived I-completion functor* $\Lambda_I\colon \mathsf{D}(R) \to \mathsf{D}(R)$ is defined by

$$\Lambda_I(M) = \mathsf{R}\mathrm{Hom}_R(K_\infty(I), M).$$

An object $M \in \mathsf{D}(R)$ is *derived I-complete* if the canonical map $M \to \Lambda_I M$ is an equivalence. We denote the full subcategory of derived I-torsion objects by $\mathsf{D}(R)^{I\text{-tors}}$.

Remark 6.11. It is worth commenting on why one should view Λ_I as some sort of completion functor. Recall that $K_\infty(x)$ is the homotopy colimit of $K(x^i)$, and therefore $\Lambda_I M$ is the homotopy limit of $\mathsf{R}\mathrm{Hom}_R(K(x^i), M)$. The unstable Koszul complexes are self-dual and compact, so this is equivalent to the homotopy limit of $K(x^i) \otimes^{\mathsf{L}}_R M$ up to shift. As we have seen above $K(x^i)$ may be considered as closely related to the quotient R/x^i, so we are taking a homotopy limit of a system which is very similar to $R/x^i \otimes^{\mathsf{L}}_R M$, which should be reminiscent of the definition of the completion.

From this definition, we can make two easy observations.

Theorem 6.12. *Let R be a commutative ring and I be an ideal generated by a finite sequence.*

(1) Greenlees-May duality*: The endofunctors Γ_I and Λ_I form an adjunction.*
(2) MGM equivalence*: The natural maps $\Gamma_I M \to \Gamma_I\Lambda_I M$ and $\Lambda_I\Gamma_I M \to \Lambda_I M$ are equivalences for all $M \in \mathsf{D}(R)$. Consequently the functors*

$$\Lambda_I : \mathsf{D}(R)^{I\text{-tors}} \rightleftarrows \mathsf{D}(R)^{I\text{-comp}} : \Gamma_I$$

give an equivalence of categories.

Proof. Item (1) is an immediate consequence of the tensor-hom adjunction.

For (2), note that $\Gamma_I \Lambda_I M \to \Gamma_I M$ is an equivalence if and only if $\mathsf{R}\mathrm{Hom}_R(K(I), M) \to \mathsf{R}\mathrm{Hom}_R(K(I), \Lambda M)$ is an equivalence by Corollary 6.5. This is true by applying GM-duality on the right hand side, and using the fact that $K(I)$ is derived I-torsion. The proof that the other map is an equivalence is similar. □

Remark 6.13. Here we defined derived completion in terms of derived torsion. We could have also given a different definition of derived I-completion (at least in the Noetherian setting) in an analogous way to Theorem 5.16, by using the left derived functors of the I-adic completion functor. However there are significant technicalities to overcome if one chooses this path, as I-adic completion is neither left nor right exact. For example, it is not true in general that for an R-module M we have $H_0(\Lambda_I M) = M_I^\wedge$.

Corollary 6.14. *The derived completion functor* $\Lambda_I \colon \mathsf{D}(R) \to \mathsf{D}(R)$ *is idempotent.*

Proof. This follows immediately from the MGM equivalence, or alternatively, since $\Lambda_I M \to \Lambda_I \Lambda_I M$ is an equivalence if and only if $\mathsf{R}\mathrm{Hom}_R(K(I), M) \to \mathsf{R}\mathrm{Hom}_R(K(I), \Lambda_I(M))$ is an equivalence as verified in the proof of the previous result. □

For completeness, let us state the relationship between derived completion and ordinary completion. Firstly we need the following definition.

Definition 6.15. Let R be a commutative ring and I be an ideal of R. The *local homology modules* of R, are defined to be the left derived functors of I-adic completion:

$$H_i^I(M) = L_i((-)_I^\wedge)(M).$$

Therefore $H_i^I(M) = H_i(P_I^\wedge)$ where P is a projective resolution of M.

Remark 6.16. Since I-adic completion is neither left nor right exact in general, $H_0^I(M)$ need not be the same as $M_I^\wedge$.

The following gives an analogue to Theorem 5.16 for local homology. We will not give the proof of the following - the strategy is similar to that used in the proof of Theorem 5.16 but the technicalities are more pronounced.

Theorem 6.17. *Let R be a commutative Noetherian ring and I be an ideal of R. Then $H_i(\Lambda_I M) = H_i^I(M)$ for all R-modules M. In particular, $\Lambda_I R \simeq R_I^\wedge$.*

Remark 6.18. Note that the functors Γ_I and Λ_I exist for any commutative ring and finitely generated ideal I, and satisfy Greenlees-May duality and MGM equivalence in this setting. However, the relationship to the local (co)homology modules does not exist in this broad generality; one needs to put assumptions on R and/or I; in particular R being Noetherian suffices.

7. Localization theory in triangulated categories

In this section, we take the results of the previous section as the bedrock for introducing a formalism of local cohomology in general (tensor) triangulated categories.

7.1. **Localization and colocalization.** We have seen a few localization and colocalization functors throughout the course so far without giving them a detailed study of their own, for instance, they appeared in the proof of Proposition 3.13, in Exercise A.47, and in the previous section. In order to general the ideas of the previous section to general trianguated categories, we will now make an in-depth study of (co)localizations and record their properties.

Definition 7.1. A triangulated functor $L\colon \mathsf{T} \to \mathcal{S}$ is a *localization* if it has a fully faithful right adjoint. Note then that $\mathcal{S} \simeq L\mathsf{T}$, the essential image of L. Frequently we write L for the composite of L followed by the inclusion. A *colocalization* is the dual concept.

Remark 7.2. Some authors might call the above a *Bousfield localization* to distinguish it from other types of localizations. In this course, we will only see Bousfield localizations.

Remark 7.3. An alternative more explicit definition of a localization is as follows. A localization is an exact functor $L\colon \mathsf{T} \to \mathsf{T}$ together with a natural transformation $\eta\colon \mathrm{id} \to L$ such that $L\eta\colon L \to L^2$ is an isomorphism, and $L\eta = \eta L$. Note the difference between the codomains in the definitions.

Definition 7.4. Let L be a localization on T. An object $X \in \mathsf{T}$ is said to be L-local if the unit map $X \to LX$ is an isomorphism, and is said to be L-acyclic if $LX \simeq 0$.

With the definitions set, we may now turn to proving various properties of (co)localizations. Firstly, we show that localizations and colocalizations always come in pairs.

Proposition 7.5. *Let T be a triangulated category. There is a one-to-one correspondences between localizations of T and colocalizations of T, in which L and Γ correspond if and only if $\Gamma X \to X \to LX \to \Sigma\Gamma X$ is a triangle for all $X \in \mathsf{T}$.*

Proof. Suppose that L is a localization of T. By (TR1) we may extend the unit map $X \to LX$ to give a triangle

$$\Gamma X \to X \to LX \to \Sigma\Gamma X.$$

By applying the triangulated functor L and using Proposition 2.18 we see that $L\Gamma X \simeq 0$, so ΓX is L-acyclic. Conversely, suppose that Z is L-acyclic. Then $\mathrm{Hom}_{\mathsf{T}}(Z, LX) = 0$ since $\mathrm{Hom}_{\mathsf{T}}(Z, LX) = \mathrm{Hom}_{\mathsf{T}}(LZ, LX) = 0$ by adjunction. Therefore, applying $\mathrm{Hom}_{\mathsf{T}}(Z, -)$ to the above triangle, we see that $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) = \mathrm{Hom}_{\mathsf{T}}(Z, X)$. Therefore ΓX has a universal property: if Z is L-acyclic and there is a map $f\colon Z \to X$, then f factors as $Z \to \Gamma X \to X$.

Normally applying (TR3) to construct maps is not functorial, but the above universal property of ΓX allows us to make Γ functorial as we now describe. Let $f\colon A \to X$ be a morphism in T. By (TR3) we obtain a map $\Gamma(f)$ in the commutative diagram

$$\begin{array}{ccccccc} \Sigma^{-1}LA & \longrightarrow & \Gamma A & \longrightarrow & A & \longrightarrow & LA \\ \downarrow & & \downarrow{\scriptstyle \Gamma(f)} & & \downarrow & & \downarrow \\ \Sigma^{-1}LX & \longrightarrow & \Gamma X & \longrightarrow & X & \longrightarrow & LX \end{array}$$

So to see that Γ is functorial, it suffices to show that $\Gamma(f)$ is unique which is immediate from the universal property. Therefore, given a localization L, we have constructed a functor $\Gamma\colon \mathsf{T} \to \mathsf{T}$.

It remains to prove that Γ is triangulated and is a colocalization. Firstly, we consider the diagram

$$\begin{array}{ccccccc}
LX & \longrightarrow & \Sigma\Gamma X & \longrightarrow & \Sigma X & \longrightarrow & \Sigma LX \\
\downarrow{\scriptstyle \sim} & & \vdots & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \sim} \\
\Sigma^{-1}L\Sigma X & \longrightarrow & \Gamma\Sigma X & \longrightarrow & \Sigma X & \longrightarrow & L\Sigma X
\end{array}$$

in which the top row is the triangle obtained by applying Σ to the standard triangle, and the bottom row is the standard triangle on ΣX. Using (TR3) and the 5-lemma, we deduce that $\Gamma\Sigma \simeq \Sigma\Gamma$. We use this isomorphism implicitly in what follows.

Consider a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. We may complete the map Γf to a triangle by (TR1). Consider the diagram

$$\begin{array}{ccccccc}
\Gamma X & \xrightarrow{\Gamma f} & \Gamma Y & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & \Sigma\Gamma X \\
\downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_Y} & & \vdots{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \Sigma\epsilon_X} \\
X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & \Sigma X
\end{array}$$

in which the dashed map exists by (TR3). Suppose that W is L-acyclic. By applying $\mathrm{Hom}_{\mathsf{T}}(W, -)$ and the 5-lemma, we see that $\mathrm{Hom}_{\mathsf{T}}(W, C) \to \mathrm{Hom}_{\mathsf{T}}(W, Z)$ is an isomorphism. Therefore C is terminal amongst L-acyclic objects mapping to Z. We proved above that ΓZ admits the same universal property, and hence $C \simeq \Gamma Z$. As in the proof of functoriality above, one then identifies α and β with Γg and Γh. Therefore Γ is triangulated as required.

It remains to prove that Γ is a colocalization. Consider the inclusion of the L-acyclic objects into T. We have seen that for any L-acyclic Z, we have that $\mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X) \to \mathrm{Hom}_{\mathsf{T}}(Z, X)$ is an isomorphism. Therefore Γ is right adjoint to this inclusion functor. To see that the inclusion is fully faithful, we just need to see that the counit is an isomorphism, i.e., that the map $\epsilon \colon \Gamma Z \to Z$ is an isomorphism if Z is L-acyclic. This is immediate by the standard triangle. □

Next, we give various equivalent characterisations of local objects. We leave the dual colocal version to the reader. We use the following notation: for any subcategory $\mathcal{X}$,

$$\mathcal{X}^{\perp} = \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}.$$

Proposition 7.6. *Let T be a triangulated category and L be a localization. Write Γ for corresponding colocalization as in Proposition 7.5. The following are equivalent for an object $X \in \mathsf{T}$:*

(1) X is L-local;
(2) $X \simeq LY$ for some $Y \in \mathsf{T}$;
(3) $\mathrm{Hom}_{\mathsf{T}}(Z, X) \simeq 0$ for all L-acyclic objects Z;
(4) X is Γ-acyclic.

In particular, $\mathrm{im}(L) = \ker(L)^{\perp}$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (3) we have $\mathrm{Hom}_{\mathsf{T}}(Z, X) = \mathrm{Hom}_{\mathsf{T}}(Z, LY) = \mathrm{Hom}_{\mathsf{T}}(LZ, LY)$ since L is a localization. Now this is zero for all L-acyclic Z as required. For (3) $\Rightarrow$ (4), as $L\Gamma \simeq 0$ as proved in Proposition 7.5, the map $\epsilon \colon \Gamma X \to X$ is zero by assumption, and hence $\Gamma\epsilon = 0$. However, $\Gamma\epsilon \colon \Gamma^2 X \to \Gamma X$ is an isomorphism. Therefore $\Gamma X \simeq 0$ as required. Finally, the implication (4) $\Rightarrow$ (1) follows immediately from the defining triangle $\Gamma X \to X \to LX \to \Sigma\Gamma X$ by Proposition 2.18. □

Corollary 7.7. *Let L be a localization. Then the full subcategory $\ker(L)$ of L-acyclics is localizing, and the full subcategory $\mathrm{im}(L)$ of L-local objects is colocalizing.*

Proof. Since L is a triangulated functor, $\ker(L)$ is thick. Now $X \in \ker(L)$ if and only if $\mathrm{Hom}_{\mathsf{T}}(X, Y) = 0$ for all L-local objects L. Therefore $\ker(L)$ is closed under coproducts, and hence is localizing. The argument that $\mathrm{im}(L)$ is colocalizing is similar. □

7.2. **Tensor versions and smashing localizations.** In this subsection we consider localization functors on tensor-triangulated categories. In order to ensure good behaviour with the tensor product, it is necessary to refine the notion of localization slightly.

Definition 7.8. A localization functor $L\colon \mathsf{T} \to \mathsf{T}$ is said to be *monoidal* if $LX \simeq 0$ implies that $L(X \otimes Y) \simeq 0$ for all $Y \in \mathsf{T}$. A colocalization functor $\Gamma\colon \mathsf{T} \to \mathsf{T}$ is *monoidal* if $\Gamma X \simeq 0$ implies that $\Gamma F(Y, X) \simeq 0$ for all $Y \in \mathsf{T}$.

Lemma 7.9. *Let $L\colon \mathsf{T} \to \mathsf{T}$ be a monoidal localization. There is a natural map $\alpha_X\colon L\mathbb{1} \otimes X \to LX$ for all $X \in \mathsf{T}$.*

Proof. Applying L to the defining triangle we have a triangle

$$L(\Gamma\mathbb{1} \otimes X) \to LX \to L(L\mathbb{1} \otimes X) \to \Sigma L(\Gamma\mathbb{1} \otimes X).$$

The first entry is 0 since L is monoidal, and hence $LX \to L(L\mathbb{1} \otimes X)$ is an isomorphism by Proposition 2.18. Therefore we have a natural map $L\mathbb{1} \otimes X \to L(L\mathbb{1} \otimes X) \simeq LX$ as required. □

Proposition 7.10. *Let T be a tensor-triangulated category, and let L be a monoidal localization. Write $\mathcal{S}$ for the full subcategory of L-local objects. The following are equivalent:*

(i) The natural map $\alpha_X\colon L\mathbb{1} \otimes X \to LX$ from Lemma 7.9 is an equivalence for all $X \in \mathsf{T}$.
(ii) $i\colon \mathcal{S} \to \mathsf{T}$ preserves coproducts (equivalently, $L\colon \mathsf{T} \to \mathsf{T}$ preserves coproducts).
(iii) $\mathcal{S}$ is a localizing subcategory of T.

If any of these equivalent conditions hold, we say that L is smashing.

Proof. This is left as Exercise A.47. □

We also introduce another well behaved form of localization.

Definition 7.11. A localization functor L is *finite* if $\ker(L)$ is generated by compact objects.

We now have an existence result for finite localizations.

Theorem 7.12. *Let T be a tensor-triangulated category with coproducts which is compactly generated by rigid objects, and suppose that $\mathcal{K}$ is a set of compact objects in T. There is a smashing localization functor $L_{\mathcal{K}}$ which depends only on $\mathsf{Thick}^{\otimes}(\mathcal{K})$, such that $\ker(L_{\mathcal{K}}) = \mathsf{Loc}^{\otimes}(\mathcal{K})$.*

Proof. Write $\mathcal{J}$ for the union of the sets of objects of the form $K \otimes G_1 \otimes G_2 \otimes \cdots G_n$ where $K \in \mathcal{K}$, each $G_i \in \mathcal{G}$ where $\mathcal{G}$ is a set of compact and rigid generators for T, and $n \in \mathbb{N}$. Firstly, we have

$$\mathsf{Thick}^{\otimes}(\mathcal{K}) = \mathsf{Thick}(\mathcal{J})$$

since the left hand side contains $\mathcal{J}$, and the right hand side is a thick $\otimes$-ideal. Therefore every object in $\mathsf{Thick}^{\otimes}(\mathcal{K})$ is compact (by Exercise A.24). In a similar way we have $\mathsf{Loc}^{\otimes}(\mathcal{K}) = \mathsf{Loc}(\mathcal{J})$.

By applying Construction 3.20 and Theorem 3.21 to the set $\mathcal{J}$ of compact objects we have for each $X \in \mathsf{T}$, a triangle $\Gamma X \to X \to LX$ such that $\Gamma X \in \mathsf{Loc}^{\otimes}(\mathcal{K})$ and $\mathrm{Hom}_{\mathsf{T}}(Z, LX) = 0$ for all $Z \in \mathsf{Loc}^{\otimes}(\mathcal{K})$. The same proof as Proposition 7.5 shows that Γ can be made into a colocalization.

Note that $\Gamma X \simeq 0$ if and only if $\mathrm{Hom}_{\mathsf{T}}(Z, X) = 0$ for all $Z \in \mathsf{Loc}^{\otimes}(\mathcal{K})$ as $\mathrm{Hom}_{\mathsf{T}}(Z, X) = \mathrm{Hom}_{\mathsf{T}}(Z, \Gamma X)$. So if $\Gamma X \simeq 0$, then $\mathrm{Hom}_{\mathsf{T}}(Z, F(Y, X)) = \mathrm{Hom}_{\mathsf{T}}(Y, F(Z, X)) = 0$ and hence $\Gamma F(Y, X) \simeq 0$ for all $Y \in \mathsf{T}$ so that Γ is monoidal. By Exercise A.45, it follows that L is monoidal. Now X is L-local if and only if $\mathrm{Hom}_{\mathsf{T}}(Z, X) = 0$ for all $Z \in \mathsf{Thick}^{\otimes}(\mathcal{K})$. Since any such Z is compact, we see that the category of L-local objects is closed under coproducts, and hence L is smashing by Proposition 7.10.

Finally it remains to see that $\ker(L) = \mathsf{Loc}^{\otimes}(\mathcal{K})$. Note that $\ker(L)$ is localizing by Corollary 7.7 and contains $\mathcal{J}$. Therefore $\mathsf{Loc}^{\otimes}(\mathcal{K}) \subseteq \ker(L)$. Conversely, if $LX \simeq 0$, then $X \simeq \Gamma X \in \mathsf{Loc}^{\otimes}(\mathcal{K})$. ☐

Remark 7.13. If T were rigidly-compactly generated, then in the first paragraph of the previous proof it is not necessary to take iterated tensor products when defining $\mathcal{J}$. Indeed, in this setting $\mathsf{Thick}^{\otimes}(\mathcal{K}) = \mathsf{Thick}(\mathcal{K} \otimes \mathcal{G})$.

8. Duality theorems

In this section we reach the culmination of this course. Firstly, we will use the theory of localizations to show that any set of compact objects in a nice enough tensor-triangulated category T gives rise to two localization and two colocalization functors which satisfy strong compatibility results, in particular a form of local duality. We will then see that in the case of $\mathsf{D}(R)$ we recover local cohomology and Grothendieck local duality from this very general theory.

8.1. **The abstract story.** We apply the localization theory set up in the previous section to produce (derived) torsion and completion functors in wide generality.

Definition 8.1. Let T be a tensor-triangulated category which is compactly generated by rigid objects, and let $\mathcal{K}$ be a set of compact objects in T. We refer to the pair $(\mathsf{T}, \mathcal{K})$ as a *local duality context.*

Recall that for a subcategory $\mathcal{X}$ of T we write

$$\mathcal{X}^{\perp} = \{Y \in \mathsf{T} \mid \mathrm{Hom}(X, Y) \simeq 0 \text{ for all } X \in \mathcal{X}\}.$$

Definition 8.2. Associated to a local duality context, we have the following full subcategories of T:

- $\mathsf{T}^{\mathcal{K}\text{-tors}} := \mathrm{Loc}^{\otimes}(\mathcal{K})$
- $\mathsf{T}^{\mathcal{K}\text{-loc}} := (\mathsf{T}^{\mathcal{K}\text{-tors}})^{\perp}$
- $\mathsf{T}^{\mathcal{K}\text{-comp}} := (\mathsf{T}^{\mathcal{K}\text{-loc}})^{\perp}$.

We call the objects in these categories $\mathcal{K}$-torsion, $\mathcal{K}$-local, and $\mathcal{K}$-complete respectively.

Remark 8.3. Note that the previous definitions only depend on $\mathcal{K}$ up to its thick tensor closure.

Theorem 8.4. *Let $(\mathsf{T}, \mathcal{K})$ be a local duality context. There are monoidal localization and colocalization functors:*

$$\Gamma\colon \mathsf{T} \to \mathsf{T}^{\mathcal{K}\text{-tors}} \quad L\colon \mathsf{T} \to \mathsf{T}^{\mathcal{K}\text{-loc}} \quad V\colon \mathsf{T} \to \mathsf{T}^{\mathcal{K}\text{-loc}} \quad \Lambda\colon \mathsf{T} \to \mathsf{T}^{\mathcal{K}\text{-comp}}$$

fitting into triangles $\Gamma X \to X \to LX$ *and* $VX \to X \to \Lambda X$*. These satisfy the following properties:*

1. *We have* $\ker(L) = \mathrm{im}(\Gamma) = \mathsf{T}^{\mathcal{K}\text{-tors}}$*;*
2. *We have* $\ker(\Gamma) = \mathrm{im}(L) = \mathrm{im}(V) = \ker(\Lambda) = \mathsf{T}^{\mathcal{K}\text{-loc}}$*;*
3. *We have* $\ker(V) = \mathrm{im}(\Lambda) = \mathsf{T}^{\mathcal{K}\text{-comp}}$*.*
4. *for any* $X \in \mathsf{T}$*,* $LX \simeq L\mathbb{1} \otimes X$ *and* $\Gamma X \simeq \Gamma\mathbb{1} \otimes X$*;*
5. *for all* $X \in \mathsf{T}$*, the natural maps* $\Gamma X \to \Gamma\Lambda X$ *and* $\Lambda\Gamma X \to \Lambda X$ *are isomorphisms;*
6. *the functors* Γ *and* Λ *give an equivalence of categories* $\mathsf{T}^{\mathcal{K}\text{-tors}} \simeq \mathsf{T}^{\mathcal{K}\text{-comp}}$*.*

Proof. By Theorem 7.12, there exists a smashing localization L and a corresponding colocalization Γ such that $\ker(L) = \mathrm{im}(\Gamma) = \mathsf{Loc}^{\otimes}(\mathcal{K})$. We define $V = F(L\mathbb{1}, -)$ and $\Lambda = F(\Gamma\mathbb{1}, -)$. These are triangulated functors which are idempotent. We will verify that these are (co)localizations later on.

Proof of (1): This was already proved in the previous paragraph.

Proof of (4): L is a smashing localization by Theorem 7.12 so $LX \simeq L\mathbb{1} \otimes X$ for all $X \in \mathsf{T}$. By (TR3) and the 5 lemma, it follows that $\Gamma X \simeq \Gamma\mathbb{1} \otimes X$ for all $X \in \mathsf{T}$.

Proof of (2): We have $\mathrm{im}(L) = \ker(\Gamma) = \mathsf{T}^{\mathcal{K}\text{-loc}}$ by Proposition 7.6. By the idempotence of V together with the associated triangle $VX \to X \to \Lambda X$, we also see that $\ker(\Lambda) = \mathrm{im}(V)$. So it remains to prove that $\ker(\Lambda) = \mathsf{T}^{\mathcal{K}\text{-loc}}$.

If $X \in \mathsf{T}^{\mathcal{K}\text{-loc}}$ then $\mathrm{Hom}(\Gamma\mathbb{1} \otimes G, X) \simeq 0$ for all $G \in \mathcal{G}$ by (1). Therefore $\Lambda\mathbb{1} \simeq F(\Gamma\mathbb{1}, X) \simeq 0$. Conversely, suppose that $\Lambda X \simeq 0$ and $Y \in \mathsf{Loc}^{\otimes}(\mathcal{K})$. Then $\mathrm{Hom}_{\mathsf{T}}(Y, X) = \mathrm{Hom}_{\mathsf{T}}(\Gamma Y, X)$ by (1). Since Γ is smashing, this is equal to $\mathrm{Hom}_{\mathsf{T}}(\Gamma\mathbb{1} \otimes Y, X) \simeq \mathrm{Hom}_{\mathsf{T}}(Y, \Lambda X) \simeq 0$.

Proof of (3): This follows the same strategy as the second paragraph of (2).

Proof of (5): We have $\Lambda LX \simeq 0$ for all X by (2). Therefore by applying Λ to the triangle $\Gamma X \to X \to LX$ we conclude that $\Lambda\Gamma X \to \Lambda X$ is an isomorphism by Proposition 2.18. The other isomorphism is similar.

Proof of (6): This is immediate from (1), (3), and (5).

Finally we argue that Λ is a localization functor. To do this, it suffices to show that the canonical map $\mathrm{Hom}_{\mathsf{T}}(\Lambda X, \Lambda Y) \to \mathrm{Hom}_{\mathsf{T}}(X, \Lambda Y)$ is an equivalence. This is equivalent to $\mathrm{Hom}_{\mathsf{T}}(VX, \Lambda Y) \simeq 0$. Now by (2), $VX \in \mathsf{T}^{\mathcal{K}\text{-loc}}$, and by (3) $\Lambda X \in \mathsf{T}^{\mathcal{K}\text{-comp}}$ so the claim follows. Finally, we need that show that Λ is moreover a monoidal localization. To see this, if $\Lambda X \simeq 0$, then $\Gamma X \simeq 0$ by (5). Again by (5) together with (4), $\Lambda(X \otimes Y) \simeq \Lambda\Gamma(X \otimes Y) \simeq \Lambda(\Gamma X \otimes Y) \simeq 0$ as required. □

Remark 8.5. If one is ever considering two different local duality contexts, we record the subscript of the set of compact objects on the associated functors, e.g., $\Gamma_{\mathcal{K}}$.

Corollary 8.6 (Greenlees-May duality)**.** *Let* $(\mathsf{T}, \mathcal{K})$ *be a local duality context. Then*

$$\Gamma\colon \mathsf{T} \rightleftarrows \mathsf{T}\colon \Lambda$$

form an adjoint pair which internalises, i.e., for all $X, Y \in \mathsf{T}$*,*

$$F(\Gamma X, Y) \simeq F(X, \Lambda Y).$$

In particular, $\Lambda X \simeq F(\Gamma\mathbb{1}, X)$*.*

Proof. We have the following string of equivalences:

$$\begin{aligned} F(\Gamma X, Y) &\simeq F(\Gamma X, \Gamma Y) && \text{as } \Gamma \text{ is a colocalization} \\ &\simeq F(\Lambda\Gamma X, \Lambda\Gamma Y) && \text{by Theorem 8.4(6)} \\ &\simeq F(\Lambda X, \Lambda Y) && \text{by the MGM equivalence (Theorem 8.4(5))} \\ &\simeq F(X, \Lambda Y) && \text{as } \Lambda \text{ is a localization.} \end{aligned}$$

The final claim holds by taking $X = \mathbb{1}$. □

Remark 8.7. Note that applying $\mathrm{Hom}_{\mathsf{T}}(\mathbb{1}, -)$ to the internalised adjunction yields the external form of the adjunction $\mathrm{Hom}_{\mathsf{T}}(\Gamma X, Y) \simeq \mathrm{Hom}_{\mathsf{T}}(X, \Lambda Y)$

Example 8.8. Let R be a commutative ring and I be an ideal generated by a finite sequence. Consider the local duality context $(\mathsf{D}(R), \{K(I)\})$. Then $\Gamma = \Gamma_I$ and $\Lambda = \Lambda_I$ by Theorem 6.6.

The above framework is a powerful setting which gives rise to various duality statements. Above we already saw Greenlees-May duality, in the exercises you will see affine duality, and here we discuss Warwick duality.

Theorem 8.9 (Warwick duality)**.** *Let* $(\mathsf{T}, \mathcal{K})$ *be a local duality context. Then there is an equivalence* $L\Lambda X \simeq \Sigma V \Gamma X$ *for all* $X \in \mathsf{T}$*.*

Proof. This is a composite of three equivalences:

$$L\Lambda X \leftarrow VL\Lambda X \rightarrow V\Sigma\Gamma\Lambda X \leftarrow V\Sigma\Gamma X.$$

For the first one, by the triangle relating V to Λ, it suffices to check that $\Lambda L \Lambda X \simeq 0$ by Proposition 2.18, which is true by Theorem 8.4(2). To verify the second equivalence, it suffices to check that $V\Lambda X \simeq 0$ by Proposition 2.18, which holds by Theorem 8.4(3). The third equivalence is the MGM equivalence (Theorem 8.4(5)). Finally note that as V is triangulated, it commutes with the shift functor. □

Example 8.10. Specialising to the case $\mathsf{T} = \mathsf{D}(\mathbb{Z})$ and $\mathcal{K} = \{\mathbb{Z}/p\}$, Warwick duality states that $\mathbb{Z}[1/p] \otimes^{\mathsf{L}}_{\mathbb{Z}} \mathbb{Z}_p^{\wedge} \simeq \Sigma \mathsf{R}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], \mathbb{Z}/p^{\infty})$.

8.2. **Recovering Grothendieck local duality.** In this section we explain how the formalism developed in the previous section (in particular, Greenlees-May duality) may be used to recover Grothendieck local duality as stated in Theorem 5.17. One should not think that the point of this theory was to give a snazzy new proof of Grothendieck local duality; rather, the triangulated formalism applies in much wider contexts: in representation theory, in algebraic geometry, and in topology.

Theorem 8.11 (Grothendieck local duality)**.** *Let* $(R, \mathfrak{m}, k)$ *be a local Gorenstein ring. Then*

$$\mathrm{Ext}^i_R(M, R^{\wedge}_{\mathfrak{m}}) = H^{\dim(R)-i}_{\mathfrak{m}}(M)^{\vee}$$

for all R*-modules* M*.*

Proof. We have the following chain of equivalences in $\mathsf{D}(R)$:

$$\begin{aligned}
\mathsf{R}\mathrm{Hom}_R(M, R^\wedge_\mathfrak{m}) &\simeq \mathsf{R}\mathrm{Hom}_R(M, \Lambda_\mathfrak{m} R) && \text{as } \Lambda_\mathfrak{m}(R) \simeq R^\wedge_\mathfrak{m} \text{ by Theorem 6.17} \\
&\simeq \mathsf{R}\mathrm{Hom}_R(\Gamma_\mathfrak{m} M, R) && \text{by Greenlees-May duality (Corollary 8.6)} \\
&\simeq \mathsf{R}\mathrm{Hom}_R(\Gamma_\mathfrak{m} M, \Gamma_\mathfrak{m} R) && \text{as } \Gamma_\mathfrak{m} \text{ is a colocalization} \\
&\simeq \mathsf{R}\mathrm{Hom}_R(\Gamma_\mathfrak{m} M, \Sigma^{-\dim(R)} E(k)) && \text{as } R \text{ is Gorenstein} \\
&\simeq \Sigma^{-\dim(R)}\mathrm{Hom}_R(\Gamma_\mathfrak{m} M, E(k)) && \text{as } E(k) \text{ is injective.}
\end{aligned}$$

Applying H_{-i} we obtain

$$\begin{aligned}
\mathrm{Ext}^i_R(M, R^\wedge_\mathfrak{m}) &= H_{\dim(R)-i}(\mathrm{Hom}_R(\Gamma_\mathfrak{m} M, E(k))) && \text{by the above equivalences} \\
&= \mathrm{Hom}_R(H_{i-\dim(R)}(\Gamma_\mathfrak{m} M), E(k)) && \text{by Exercise A.12} \\
&= \mathrm{Hom}_R(H^{\dim(R)-i}_\mathfrak{m}(M), E(k)) && \text{by Theorem 5.16} \\
&= H^{\dim(R)-i}_\mathfrak{m}(M)^\vee && \text{by definition of } (-)^\vee
\end{aligned}$$

as required. □

8.3. **The local-to-global principle.** As another application of the formalism developed in this section, we prove that $\mathsf{D}(R)$ (for R commutative Noetherian) satisfies the *local-to-global principle*, meaning that any object of $\mathsf{D}(R)$ can be recovered (in the localizing sense) from a certain collection of small objects.

In order to state and prove this result, we need to set some notation. Recall that for a commutative ring R, $\mathrm{Spec}(R)$ is the set of all prime ideals $\mathfrak{p}$ of R. Given a subset V of $\mathrm{Spec}(R)$, we write Γ_V, L_V etc., for the functors associated to the local duality context $(\mathsf{D}(R), \{K(\mathfrak{p}) \mid \mathfrak{p} \in V\})$. We set $\mathcal{V}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ and $Z(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$.

Throughout this section, we use repeatedly that in $\mathsf{D}(R)$ any localizing subcategory is a localizing $\otimes$-ideal, since $\mathsf{D}(R)$ is generated by its tensor unit. Firstly we turn to identifying the functors associated to these local duality contexts.

Lemma 8.12. *Let R be a commutative Noetherian ring. The functor $\Gamma_{\mathcal{V}(\mathfrak{p})}$ may be identified with the functor $\Gamma_\mathfrak{p}$ from Theorem 6.6.*

Proof. It suffices to show that $\mathrm{Loc}(K(\mathfrak{q}) \mid \mathfrak{p} \subseteq \mathfrak{q}) = \mathrm{Loc}(K(\mathfrak{p}))$. The reverse inclusion is clear, so we prove the forward inclusion; that is, we must prove that if $\mathfrak{p} \subseteq \mathfrak{q}$, then $K(\mathfrak{q}) \in \mathrm{Loc}(K(\mathfrak{p}))$. Since $K_\infty(\mathfrak{p})$ is dg-flat, one calculates that $K_\infty(\mathfrak{p}) \otimes^{\mathsf{L}}_R R/\mathfrak{q} \simeq R/\mathfrak{q}$. Therefore by Theorem 6.3, we also have $K_\infty(\mathfrak{p}) \otimes^{\mathsf{L}}_R K(\mathfrak{q}) \simeq K(\mathfrak{q})$, and hence $K(\mathfrak{q}) \in \mathrm{Loc}(K_\infty(\mathfrak{p})) = \mathrm{Loc}(K(\mathfrak{p}))$. □

Lemma 8.13. *Let T be a tensor-triangulated category which is compactly generated by rigid objects. Suppose that $\mathcal{K}$ and $\mathcal{L}$ are sets of compact objects of T, and $\mathrm{Loc}^\otimes(\mathcal{K}) \subseteq \mathrm{Loc}^\otimes(\mathcal{L})$. Then*

$$\Gamma_\mathcal{K} \simeq \Gamma_\mathcal{K}\Gamma_\mathcal{L} \simeq \Gamma_\mathcal{L}\Gamma_\mathcal{K}.$$

Proof. This is left as Exercise A.51. □

Lemma 8.14. *Let R be a commutative Noetherian ring, and let $\mathfrak{p}$ be a prime ideal in R. Then*

$$\mathrm{Hom}_{\mathsf{D}(R)}(R, L_{Z(\mathfrak{p})} M) \simeq \mathrm{Hom}_{\mathsf{D}(R)}(R, M)_\mathfrak{p}$$

for all $M \in \mathsf{D}(R)$.

Proof. The functor $\mathrm{Hom}_{\mathsf{D}(R)}(R,-)_{\mathfrak{p}}\colon \mathsf{D}(R)\to \mathsf{Ab}$ is homological and coproduct preserving. For all $x\in R\backslash\mathfrak{p}$, we have $\mathrm{Hom}_{\mathsf{D}(R)}(R,K(x))_{\mathfrak{p}}=0$. By a localizing subcategory argument, we therefore have $\mathrm{Hom}_{\mathsf{D}(R)}(R,\Gamma_{Z(\mathfrak{p})}M)_{\mathfrak{p}}=0$ and hence

$$\mathrm{Hom}_{\mathsf{D}(R)}(R,M)_{\mathfrak{p}}\xrightarrow{\sim}\mathrm{Hom}_{\mathsf{D}(R)}(R,L_{Z(\mathfrak{p})}M)_{\mathfrak{p}}.$$

Now applying $\mathrm{Hom}_{\mathsf{D}(R)}(-,L_{Z(\mathfrak{p})}M)$ to the triangle

$$R\xrightarrow{\cdot x}R\to K(x)$$

shows that multiplication by x is an isomorphism on $\mathrm{Hom}_{\mathsf{D}(R)}(R,L_{Z(\mathfrak{p})}M)$ if $x\in R\backslash\mathfrak{p}$, since $\mathrm{Hom}_{\mathsf{D}(R)}(K(x),L_{Z(\mathfrak{p})}M)\simeq 0$ as $K(x)$ is $Z(\mathfrak{p})$-torsion. Therefore

$$\mathrm{Hom}_{\mathsf{D}(R)}(R,L_{Z(\mathfrak{p})}M)_{\mathfrak{p}}\simeq\mathrm{Hom}_{\mathsf{D}(R)}(R,L_{Z(\mathfrak{p})}M)$$

and hence $\mathrm{Hom}_{\mathsf{D}(R)}(R,L_{Z(\mathfrak{p})}M)=\mathrm{Hom}_{\mathsf{D}(R)}(R,M)_{\mathfrak{p}}$ as desired. □

In light of the previous result, we write $L_{\mathfrak{p}}$ for the functor $L_{Z(\mathfrak{p})}$. We now have the necessary ingredients to prove the local-to-global principle for $\mathsf{D}(R)$.

Theorem 8.15 (Local-to-global principle)**.** *Let R be a commutative Noetherian ring. For any $M\in\mathsf{D}(R)$ we have*

$$\mathrm{Loc}(M)=\mathrm{Loc}(\Gamma_{\mathfrak{p}}L_{\mathfrak{p}}M\mid\mathfrak{p}\in\mathrm{Spec}(R)).$$

Proof. Note that it suffices to prove that $R\in\mathrm{Loc}(\Gamma_{\mathfrak{p}}L_{\mathfrak{p}}R\mid\mathfrak{p}\in\mathrm{Spec}(R))$ since then applying $-\otimes^{\mathsf{L}}_R M$, we obtain the desired result since $\Gamma_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ are smashing by Theorem 8.4(4). We split the proof of this up into steps.

Step 1: Let $\mathsf{X}=\{\mathfrak{p}\in\mathrm{Spec}(R)\mid\Gamma_{\mathfrak{p}}R\in\mathrm{Loc}(\Gamma_{\mathfrak{p}}L_{\mathfrak{p}}R\mid\mathfrak{p}\in\mathrm{Spec}(R))\}$. We prove that if $\mathfrak{p}\in\mathsf{X}$ and $\mathfrak{p}\subseteq\mathfrak{q}$, then $\mathfrak{q}\in\mathsf{X}$.

To see this, we have $\Gamma_{\mathfrak{q}}R\simeq\Gamma_{\mathfrak{q}}\Gamma_{\mathfrak{p}}R$ by Lemma 8.13 since

$$\mathrm{Thick}(K(\mathfrak{q}))=\mathrm{Thick}(K(\mathfrak{a})\mid\mathfrak{a}\in\mathcal{V}(\mathfrak{q}))\subseteq\mathrm{Thick}(K(\mathfrak{a})\mid\mathfrak{a}\in\mathcal{V}(\mathfrak{p}))=\mathrm{Thick}(K(\mathfrak{p}))$$

by Lemma 8.12. Since Γ is smashing (Theorem 8.4(4)), this is moreover equivalent to $\Gamma_{\mathfrak{q}}R\otimes^{\mathsf{L}}_R\Gamma_{\mathfrak{p}}R$, and this is in X by assumption (since R builds $\Gamma_{\mathfrak{q}}R$).

Step 2: Let $U(\mathfrak{q})=\mathcal{V}(\mathfrak{q})\backslash\{\mathfrak{q}\}$. We claim that there is a triangle

$$\Gamma_{U(\mathfrak{q})}R\to\Gamma_{\mathfrak{q}}R\to\Gamma_{\mathfrak{q}}L_{\mathfrak{q}}R$$

for any $\mathfrak{q}\in\mathrm{Spec}(R)$.

To prove this, note that we have a triangle $\Gamma_{U(\mathfrak{q})}M\to M\to L_{U(\mathfrak{q})}M$ for any $M\in\mathsf{D}(R)$ by Theorem 8.4. We take $M=\Gamma_{\mathfrak{q}}R$. Then $\Gamma_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}R\simeq\Gamma_{U(\mathfrak{q})}R$ by Lemma 8.13 since $U(\mathfrak{q})\subseteq\mathcal{V}(\mathfrak{q})$. So it remains to prove that $L_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}R\simeq L_{\mathfrak{q}}\Gamma_{\mathfrak{q}}$. In order to do this, we consider the triangle

$$\Gamma_{Z(\mathfrak{q})}L_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}R\to L_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}R\to L_{\mathfrak{q}}L_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}R.$$

The latter term is equivalent to $L_{\mathfrak{q}}\Gamma_{\mathfrak{q}}M$ by Lemma 8.13 as $U(\mathfrak{q})\subseteq Z(\mathfrak{q})$. For the first term, we have

$$\Gamma_{Z(\mathfrak{q})}L_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}\simeq L_{U(\mathfrak{q})}\Gamma_{Z(\mathfrak{q})}\Gamma_{\mathfrak{q}}\simeq L_{U(\mathfrak{q})}\Gamma_{Z(\mathfrak{q})\cap\mathcal{V}(\mathfrak{q})}$$

where the first equivalence follows from the functors being smashing, and the second from a similar argument to the proof of Lemma 8.13. Since $Z(\mathfrak{q})\cap\mathcal{V}(\mathfrak{q})\subseteq U(\mathfrak{q})$, we see that $L_{U(\mathfrak{q})}\Gamma_{Z(\mathfrak{q})\cap\mathcal{V}(\mathfrak{q})}\simeq 0$. Therefore from the above triangle we deduce that $L_{U(\mathfrak{q})}\Gamma_{\mathfrak{q}}R\simeq L_{\mathfrak{q}}\Gamma_{\mathfrak{q}}R$ as required.

Step 3: We now prove that $\mathsf{X} = \mathrm{Spec}(R)$.

In order to do this, we suppose not and derive a contradiction. So, if $\mathsf{X} \neq \mathrm{Spec}(R)$, then since R is Noetherian, there is a maximal $\mathfrak{q} \in \mathrm{Spec}(R) \backslash \mathsf{X}$. We have

$$\begin{aligned} \Gamma_{U(\mathfrak{q})}R &\in \mathrm{Loc}(K(\mathfrak{p}) \mid \mathfrak{p} \in U(\mathfrak{q})) && \text{by definition} \\ &= \mathrm{Loc}(\Gamma_\mathfrak{p} R \mid \mathfrak{p} \in U(\mathfrak{q})) && \text{by Theorem 6.3} \\ &\subseteq \mathrm{Loc}(\Gamma_\mathfrak{p} L_\mathfrak{p} R \mid \mathfrak{p} \in \mathrm{Spec}(R)) && \text{as } \mathfrak{q} \subsetneq \mathfrak{p},\ \mathfrak{p} \text{ is in } \mathsf{X} \text{ by maximality of } \mathfrak{q}. \end{aligned}$$

Now, by the triangle in Step 2, we must have $\Gamma_\mathfrak{q} R \in \mathsf{Loc}(\Gamma_\mathfrak{p} L_\mathfrak{p} R \mid \mathfrak{p} \in \mathrm{Spec}(R))$ since $\Gamma_{Z(\mathfrak{q})}R$ lies in there by the above, and $\Gamma_\mathfrak{q} L_\mathfrak{q} R$ is also in this localizing subcategory by definition. This means that $\mathfrak{q} \in \mathsf{X}$ which is a contradiction, and hence $\mathsf{X} = \mathrm{Spec}(R)$.

Step 4: We claim that if $V = \bigcup_{i \in I} \vee(\mathfrak{p}_i)$, then $\mathrm{Loc}(\Gamma_V R) = \mathrm{Loc}(\Gamma_{\mathfrak{p}_i} R \mid i \in I)$.

Note that $\Gamma_V \Gamma_{\mathfrak{p}_i} \simeq \Gamma_{\mathfrak{p}_i}$ by Lemma 8.13, so the reverse inclusion holds. For the other inclusion we have

$$\begin{aligned} \Gamma_V R &\in \mathrm{Loc}(K(\mathfrak{q}) \mid \mathfrak{q} \in V) && \text{by definition} \\ &= \mathrm{Loc}(K(\mathfrak{q}) \mid \mathfrak{q} \in \cup_{i \in I} \vee(\mathfrak{p}_i)) && \text{by assumption on } V \\ &= \mathrm{Loc}(K(\mathfrak{p}_i) \mid i \in I) && \text{by Lemma 8.12} \\ &= \mathrm{Loc}(\Gamma_{\mathfrak{p}_i} R \mid i \in I) && \text{by Theorem 6.3.} \end{aligned}$$

Step 5: We now put this all together to deduce that $R \in \mathrm{Loc}(\Gamma_\mathfrak{p} L_\mathfrak{p} R \mid \mathfrak{p} \in \mathrm{Spec}(R))$ as required. Choose a cover $\mathrm{Spec}(R) = \cup_{i \in I} \vee \mathfrak{p}_i$. We have $R \simeq \Gamma_{\mathrm{Spec}(R)} R$ since $L_{\mathrm{Spec}(R)} R \simeq 0$ by an argument similar to Lemma 8.14. Hence

$$R \simeq \Gamma_{\mathrm{Spec}(R)} R \in \mathrm{Loc}(\Gamma_{\mathfrak{p}_i} R \mid i \in I) \subseteq \mathrm{Loc}(\Gamma_\mathfrak{p} L_\mathfrak{p} R \mid \mathfrak{p} \in \mathrm{Spec}(R))$$

where the first inclusion is by Step 4, and the second inclusion is by Step 3. □

This gives the following powerful corollary which is sometimes called the detection property.

Corollary 8.16. *Let R be a commutative Noetherian ring. For $M \in \mathsf{D}(R)$, we have $M \simeq 0$ if and only if $\Gamma_\mathfrak{p} L_\mathfrak{p} M \simeq 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.* □

Appendix A. Exercises

Exercise A.1. Prove that in the definition of a triangulated category it is not necessary to assume that distinguished triangles are candidate triangles. In other words, prove that any 'triangle' satisfying (TR0)-(TR3) is necessarily a candidate triangle.

Exercise A.2. This exercise is designed to convince you that signs can be important in triangulated categories (e.g. in (TR2)). Consider the multiplication by 3 map on $\mathbb{Z}$, and the cone of this map $C(\cdot 3)$. Consider the diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & \longrightarrow & C(\cdot 3) & \longrightarrow & \Sigma\mathbb{Z} & \xrightarrow{\Sigma(\cdot 3)} & \Sigma\mathbb{Z} \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} \\
\mathbb{Z} & \longrightarrow & C(\cdot 3) & \longrightarrow & \Sigma\mathbb{Z} & \xrightarrow[-\Sigma(\cdot 3)]{} & \Sigma\mathbb{Z}
\end{array}$$

Show that for no such map f making the diagram commute (in $\mathsf{K}(\mathbb{Z})$) can exist.

Exercise A.3. Let T be a pretriangulated category.

(1) Suppose that $X \to Y \to Z \to \Sigma X$ and $X' \to Y' \to Z' \to \Sigma Z'$ are candidate triangles. Show that if their sum is a distinguished triangle, then so is each summand.
(2) Show that for any map $\theta\colon X \to Y$ in T, θ is an isomorphism if and only if there is a triangle $X \xrightarrow{\theta} Y \to 0 \to \Sigma X$.
(3) Show any triangle of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{0} \Sigma X$ is split, that is, isomorphic to a triangle of the form $X \to X \oplus Z \to Z \to \Sigma X$.

Exercise A.4. Let $F\colon \mathsf{T} \to \mathsf{U}$ be a triangulated functor. Prove that the kernel of F is a triangulated subcategory of T.

Exercise A.5. Prove that $\mathsf{K}(R)$ satisfies the following dual Ore condition: given a quasi-isomorphism $s\colon X \to X'$ and any map $f\colon X \to Y$, then there exists a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\
X' & \xrightarrow[f']{} & Y'
\end{array}$$

in which s' is also a quasi-isomorphism.

Exercise A.6. Prove Lemma 2.32.

Exercise A.7. What we have called rooves so far are infact *left rooves*. In an analogous way, one may define right rooves to be diagrams

$$\begin{array}{ccc}
 & Z & \\
\nearrow & & \overset{\sim}{\nwarrow} \\
X & & Y
\end{array}$$

The Ore condition associates to any right roof a left roof. Show that this gives a bijection between equivalence classes of right and left rooves.

Exercise A.8. Consider the full subcategory $\mathsf{D}(R)_{\geq n}$ consisting of the complexes M such that $H_i(M) = 0$ for all $i < n$. Prove that $\tau_{\geq n}$ is right adjoint to the inclusion functor $\mathsf{D}(R)_{\geq n} \hookrightarrow \mathsf{D}(R)$.

Exercise A.9. Prove Lemma 2.39.

Exercise A.10. Show that there exists a map $\alpha\colon \mathbb{Z}/p \to \Sigma\mathbb{Z}/p$ in $\mathsf{D}(\mathbb{Z})$ which is not null homotopic, but which has $H_*(\alpha) = 0$.

Exercise A.11. Give an example of a quasi-isomorphism which does not have an inverse in the category of chain complexes (i.e., find a quasi-isomorphism $f\colon M \to N$ for which there can be no chain map $g\colon N \to M$ such that $H_*(f)$ and $H_*(g)$ are inverses.)

Exercise A.12. Let I be an injective R-module. Prove that for any $M \in \mathsf{D}(R)$ and $n \in \mathbb{Z}$ there is an isomorphism $H_n(\mathrm{Hom}_R(M, I)) = \mathrm{Hom}_R(H_{-n}M, I)$.

Exercise A.13. Let k be a field. Prove that the category of vector spaces over k may be given a triangulated structure in which the shift functor is the identity, and the distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$ are the exact sequences $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X \xrightarrow{f} Y$.

Exercise A.14.

(i) Show that the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ of $\mathbb{Z}$-modules does not give rise to a triangle in $\mathsf{K}(\mathbb{Z})$.

(ii) Prove that if $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a split exact sequence of R-modules, then there is a map $h\colon N \to \Sigma L$ such that $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L$ is a triangle in $\mathsf{K}(R)$.

Exercise A.15. A complex $M \in \mathsf{D}(R)$ is said to be *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projectives. Write $\mathrm{Perf}(R)$ for the full subcategory of $\mathsf{D}(R)$ consisting of the perfect complexes. Show that $\mathrm{Perf}(R)$ is a thick subcategory.

Exercise A.16. Which of the following are compact objects?

(1) $\mathbb{Z}/p \in \mathsf{D}(\mathbb{Z})$
(2) $\mathbb{Q} \in \mathsf{D}(\mathbb{Z})$
(3) $\mathbb{Q} \in \mathsf{D}(\mathbb{Q}[x])$
(4) $\mathbb{Q} \in \mathsf{D}(\mathbb{Q}[x]/x^2)$

Exercise A.17. Let $F : \mathsf{T} \rightleftarrows \mathsf{U} : G$ be an adjunction between triangulated categories.

(1) Prove that if G preserves coproducts, then F preserves compact objects.
(2) Suppose that T is compactly generated and that F preserves compacts. Show that G preserves coproducts.

Exercise A.18. Let T be a compactly generated triangulated category. A map $f\colon X \to Y$ is said to be *phantom* if the induced map $\mathrm{Hom}_{\mathsf{T}}(C, X) \to \mathrm{Hom}_{\mathsf{T}}(C, Y)$ is zero for all compacts C. Prove that a coproduct and product preserving triangulated functor preserves phantom maps.

Exercise A.19. Prove that the full subcategory of rigid objects of a tensor-triangulated category T is thick. Prove that the full subcategory of F-compact objects of T is thick.

Exercise A.20. Let T be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is rigid. Prove that the natural map

$$F(Y, \mathbb{1}) \otimes X \to F(Y, X)$$

is an equivalence for all $Y \in \mathsf{T}$.

Exercise A.21. Let X be a rigid object of T. Prove that the functor $X \otimes - \colon \mathsf{T} \to \mathsf{T}$ commutes with products.

Exercise A.22. Prove Proposition 4.23(3).

Exercise A.23. This exercise concerns the construction of Brown-Comenetz duals in tensor-triangulated categories. These are certain 'designer' objects which play an important role in stable homotopy theory. Let T be a rigidly-compactly generated tensor-triangulated category.

(1) Let $C \in \mathsf{T}^{\mathrm{c}}$. Show that there exists an object $\mathbb{I}_C \in \mathsf{T}$ such that
$$\mathrm{Hom}_{\mathsf{T}}(-, \mathbb{I}_C) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathsf{T}}(C, -), \mathbb{Q}/\mathbb{Z}).$$
(2) Define a functor $I_C \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ by $I_C(-) := F(-, \mathbb{I}_C)$. Prove that $I_C(X) \simeq F(F(C, X), \mathbb{I}_{\mathbb{1}})$.
(3) Let $X \in \mathsf{T}$. Prove that if $I_{\mathbb{1}}(X) \simeq 0$, then $X \simeq 0$. (*Hint:* Recall that $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for abelian groups, so that if $M \in \mathsf{Mod}(\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \simeq 0$, then $M \simeq 0$.)
(4) Let $\mathcal{X}$ be a set of a compact objects and suppose that if $X \in \mathcal{X}$ and $C \in \mathsf{T}^{\mathrm{c}}$, then $C \otimes X \in \mathcal{X}$. Consider the set
$$\mathcal{X}^{\perp_{\mathbb{Z}}} := \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i X, Y) \simeq 0 \text{ for all } X \in \mathcal{X} \text{ and } i \in \mathbb{Z}\}.$$
Show that if $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$, then $X \otimes Y \simeq 0$. Deduce that if $Y \in \mathcal{X}^{\perp_{\mathbb{Z}}}$, then $I_C(Y) \in \mathcal{X}^{\perp_{\mathbb{Z}}}$ for all $C \in \mathsf{T}^{\mathrm{c}}$.
(5) Consider $\mathsf{T} = \mathsf{D}(R)$ for a commutative ring R. What is $\mathbb{I}_R$?

Exercise A.24. Let T be a tensor-triangulated category.

(1) Prove that if X is compact and Y is rigid, then $X \otimes Y$ is compact.
(2) Prove that if X and Y are rigid, then $X \otimes Y$ is rigid.

Exercise A.25. Let T be a rigidly-compactly generated tensor-triangulated category, and let $\mathcal{S}$ be a thick $\otimes$-ideal of T^{c}. Prove that $\mathcal{S}$ is radical, i.e., for all $X \in \mathsf{T}^{\mathrm{c}}$, if $X^{\otimes n} \in \mathcal{S}$ for some n, then $X \in \mathcal{S}$.

Exercise A.26. Let $F \colon \mathrm{Ch}(R) \to \mathrm{Ch}(S)$ be an exact functor. Prove that there exists a natural isomorphism $\phi \colon F\Sigma \xrightarrow{\sim} \Sigma F$, and for every $f \colon M \to N$ in $\mathrm{Ch}(R)$, there is an isomorphism $\theta \colon F(C(f)) \xrightarrow{\sim} C(F(f))$ making the diagram
$$\begin{array}{ccccc}
F(N) & \xrightarrow{F(i_f)} & F(C(f)) & \xrightarrow{F(c_f)} & F(\Sigma M) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi_M} \\
F(N) & \xrightarrow[i_{F(f)}]{} & C(F(f)) & \xrightarrow[c_{F(f)}]{} & \Sigma F(M)
\end{array}$$
commute.

Exercise A.27. Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring. Prove that
$$\mathrm{Tor}_i^R(M, N)^\vee = \mathrm{Ext}_R^i(M, N^\vee)$$
for all R-modules M and N. Show that if M is moreover finitely generated, then
$$\mathrm{Ext}_R^i(M, N)^\vee = \mathrm{Tor}_i^R(M, N^\vee).$$

Exercise A.28. Prove Lemma 5.10.

Exercise A.29. Consider the case $R = \mathbb{Z}$ and $I = (p)$. Which of the following abelian groups are (p)-power torsion?

$$\mathbb{Z}/p \qquad \mathbb{Z}/p^2 \qquad \mathbb{Z}/p^\infty \qquad \mathbb{Z} \qquad \mathbb{Q} \qquad \bigoplus_{i=1}^{\infty} \mathbb{Z}/p \qquad \prod_{i=1}^{\infty} \mathbb{Z}/p \qquad \prod_{i=1}^{\infty} \mathbb{Z}/p^i$$

Exercise A.30. If $\alpha \in I$, show that $K_\infty(I, \alpha) \to K_\infty(I)$ is a quasi-isomorphism.

Exercise A.31. Calculate $H^*_{(p)}(\mathbb{Z})$ and $H^*_{(x)}(k[x])$ using the equivalence of Theorem 5.16.

Exercise A.32. Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring of dimension d. Prove that for any finitely generated R-module M, we have

$$H^i_{\mathfrak{m}}(M) = \operatorname{Ext}^{d-i}_R(M, R)^\vee.$$

Exercise A.33. Local cohomology satisfies some convenient base change properties. Recall that given a map $\theta\colon R \to S$ of commutative rings, there is an induced restriction of scalars functor $\theta^*\colon \mathsf{Mod}(S) \to \mathsf{Mod}(R)$ which has a left adjoint $S \otimes_R -\colon \mathsf{Mod}(R) \to \mathsf{Mod}(S)$. Let $\theta\colon R \to S$ be a map of commutative Noetherian rings, let I be an ideal of R, and write IS for the induced ideal of S, i.e., if $I = (x_1, \cdots, x_n)$, then $IS = (\theta(x_1), \cdots, \theta(x_n))$. Prove the following base change properties:

1. If S is a flat R-module, then for any R-module M and integer i, there is a natural isomorphism $S \otimes_R H^i_I(M) \cong H^i_{IS}(S \otimes_R M)$.
2. For any S-module N and integer i, there is a natural isomorphism $H^i_{IS}(N) \cong H^i_I(\theta^* N)$.

Exercise A.34. Cohen's structure theorem states that for any complete local Noetherian ring R there is a surjective map of rings $A \to R$ where A is a regular local ring. Using this, prove that for any complete local Noetherian ring R and finitely generated R-module M, the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are Artinian for all $i \geq 0$.

Exercise A.35. Let $(R, \mathfrak{m}, k)$ be a local ring.

1. For any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ of R, show that $H^i_{\mathfrak{m}}(-)_{\mathfrak{p}} = 0$.

As such, one cannot localize local cohomology modules. Now suppose that R is moreover Gorenstein and complete. To rectify the inability of one to localize local cohomology modules, one defines the *dual localization* at $\mathfrak{p}$ to be the functor $\mathcal{L}_{\mathfrak{p}}(M) = ((M^{\vee_{\mathfrak{m}}})_{\mathfrak{p}})^{\vee_{\mathfrak{p}}}$. That is, we first do the $\mathfrak{m}$-Matlis dual, then $\mathfrak{p}$-localize, and then take the $\mathfrak{p}$-Matlis dual.

2. Let M be a finitely generated R-module. Prove that $\operatorname{Ext}^i_R(M, N) = \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.
3. Show that $\mathcal{L}_{\mathfrak{p}}(E(R/\mathfrak{m})) = R_{\mathfrak{p}}/\mathfrak{p}$.
4. Prove that $\mathcal{L}_{\mathfrak{p}}(H^i_{\mathfrak{m}}(M)) = H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}}(M_{\mathfrak{p}})$ for every finitely generated R-module M. You may use without proof the fact that $\dim(R/\mathfrak{p}) + \dim(R_{\mathfrak{p}}) = \dim(R)$ for a Gorenstein local ring.

Exercise A.36. A local Noetherian ring $(R, \mathfrak{m}, k)$ is *Cohen-Macaulay* if $H^i_{\mathfrak{m}}(R) = 0$ for all $i \neq \dim(R)$. By using the long exact sequence in local cohomology, prove that if R is Cohen-Macaulay and x is a regular element in R, then R/x is Cohen-Macaulay.

Exercise A.37. Let R be a commutative ring and x be a regular element in R. Prove that multiplication by x is surjective on $H^1_{(x)}(R)$.

Exercise A.38. Let I be an ideal in a commutative ring R.

(1) Show that $\mathrm{Hom}_R(R/I^n, M) = \{x \in M \mid I^n x = 0\}$.
(2) For $m \leq n$, note that there is an inclusion $I^n \subseteq I^m$, and therefore we obtain a surjection $R/I^n \to R/I^m$. Therefore we have maps $\mathrm{Hom}_R(R/I^m, M) \to \mathrm{Hom}_R(R/I^n, M)$. Show that $\varinjlim \mathrm{Hom}_R(R/I^n, M) = T_I M$.

Exercise A.39. Let R be a commutative ring and $x \in R$. Show that for each $k > 1$,

$$K(x^{k-1}) \to K(x^k) \to K(x)$$

is a distinguished triangle in $\mathsf{D}(R)$.

Exercise A.40. Let R be a commutative ring. Show that if $I = (x_1, \cdots, x_n)$ is an ideal generated by a regular sequence of elements of R, then R/I is a compact object of $\mathsf{D}(R)$.

Exercise A.41. Let R be a commutative ring and $I = (x_1, \cdots, x_n)$ be a finitely generated ideal. Prove that $DK(I) \simeq \Sigma^n K(I)$.

Exercise A.42. Let $R = \mathbb{Z}$ and $I = (p)$. What are $K(I)$ and $K_\infty(I)$ in this case? You should write these as complexes concentrated in a single degree.

Exercise A.43. Let us consider the example of the integers $R = \mathbb{Z}$ and $I = (p)$ in detail.

(1) Give a closed form for the local homology modules $H_*^{(p)}(-)$ in terms of Ext-modules.
(2) Hence or otherwise, show that $H_0^{(p)}$ is not left exact in general.
(3) Was there anything special about the integers or the element p in part (1)? In other words, given a commutative Noetherian ring R and an element $x \in R$, do you need any assumptions to make an analogous statement true?

Exercise A.44. Let R be a commutative ring and I be a finitely generated ideal.

(1) Prove that if M is a derived torsion complex, then the functional dual DM is derived complete.
(2) Is the converse to (1) true?
(3) Prove that if M is compact and derived torsion, then DM is also derived torsion.

Exercise A.45. Let T be a tensor-triangulated category. Prove that the one-to-one correspondence between localizations and colocalizations in Proposition 7.5 restricts to a one-to-one correspondence between monoidal localizations and monoidal colocalizations.

Exercise A.46. Let T be a tensor-triangulated category, and L be a monoidal localization. Prove that the natural map $\alpha_X\colon L\mathbb{1} \otimes X \to LX$ from Lemma 7.9 is an isomorphism if X is rigid.

Exercise A.47. Let T be a tensor-triangulated category, L be a monoidal localization, and write $\mathcal{S}$ for the full subcategory of L-local objects.

(1) Prove Proposition 7.10.
(2) Give an example of a smashing localization.
(3) Prove that $L\colon \mathsf{T} \to \mathcal{S}$ preserve compacts if L is smashing. Does this still hold if L is not smashing?

Exercise A.48. Let $(\mathsf{T}, \mathcal{K})$ be a local duality context, and assume that $\mathcal{K} = \{K\}$ is a singleton. An object $X \in \mathsf{T}$ is said to be *homologically K-local* if $\mathrm{Hom}(Y, X) \simeq 0$ whenever $K \otimes Y \simeq 0$. A map $f\colon X \to Y$ in T is said to be a *K-equivalence* if $K \otimes f$ is an equivalence. This question is about showing that completion may be equivalently described as homological K-localization.

(1) Show that X is homologically K-local if and only if for all $f\colon Y \to Z$ such that $K \otimes f$ is an equivalence, the induced map $\mathrm{Hom}(f, X)$ is an equivalence.
(2) Prove that the map $X \to \Lambda X$ is a K-equivalence.
(3) Prove that X is complete if and only if X is homologically K-local.

Exercise A.49. Let $(\mathsf{T}, \mathcal{K})$ be a local duality context. Prove that $\Lambda F(X,Y) \simeq F(X, \Lambda Y)$ for all $X, Y \in \mathsf{T}$.

Exercise A.50. Let $(\mathsf{T}, \mathcal{K})$ be a local duality context.

(1) Prove that $\Gamma F(X,Y) \simeq F(X, \Gamma Y)$ for all $X \in \mathsf{T}^{\mathrm{c}}$ and $Y \in \mathsf{T}$.
(2) Fix an object $\Omega \in \mathsf{T}$ and define functors $D_\Omega\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ and $D_{\mathcal{K},\Omega}\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ by $D_\Omega = F(-,\Omega)$ and $D_{\mathcal{K},\Omega} = F(-, \Gamma\Omega)$. Prove that $\Lambda D_\Omega^2(X) \simeq D_{\mathcal{K},\Omega}^2(X)$ for all $X \in \mathsf{T}^{\mathrm{c}}$.
(3) Suppose that Ω is compact. Show that $\Lambda X \simeq D_{\mathcal{K},\Omega}^2(X)$ for all $X \in \mathsf{T}^{\mathrm{c}}$.

Exercise A.51. Let T be a tensor-triangulated category which is compactly generated by rigid objects. Suppose that $\mathcal{K}$ and $\mathcal{L}$ are sets of compact objects of T, and $\mathrm{Loc}^{\otimes}(\mathcal{K}) \subseteq \mathrm{Loc}^{\otimes}(\mathcal{L})$. Prove that

$$\Gamma_{\mathcal{K}} \simeq \Gamma_{\mathcal{K}}\Gamma_{\mathcal{L}} \simeq \Gamma_{\mathcal{L}}\Gamma_{\mathcal{K}}.$$

Write down and prove analogous statements for L, V and Λ.

Exercise A.52. Let R be a commutative Noetherian ring. For each $\mathfrak{p} \in \mathrm{Spec}(R)$, write $\Gamma_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ as in Section 8.3. Define a support theory on objects of $\mathsf{D}(R)$ via

$$\mathrm{supp}(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \Gamma_{\mathfrak{p}} L_{\mathfrak{p}} X \not\simeq 0\}.$$

(1) Verify that supp satisfies the following properties:
  (a) $\mathrm{supp}(\mathbb{1}) = \mathrm{Spec}(R)$ and $\mathrm{supp}(0) = \varnothing$;
  (b) $\mathrm{supp}(\Sigma M) = \mathrm{supp}(M)$ for all $M \in \mathsf{D}(R)$;
  (c) $\mathrm{supp}(M) \subseteq \mathrm{supp}(L) \cup \mathrm{supp}(N)$ for any triangle $L \to M \to N$ in $\mathsf{D}(R)$;
  (d) $\mathrm{supp}(\oplus M_i) = \cup \mathrm{supp}(M_i)$ for all $\{M_i\}$;
  (e) $\mathrm{supp}(M \otimes_R^{\mathsf{L}} N) \subseteq \mathrm{supp}(M) \cap \mathrm{supp}(N)$ for all $M, N \in \mathsf{D}(R)$.
(2) Let $V \subseteq \mathrm{Spec}(R)$. Show that $\mathrm{supp}^{-1}(V) = \{X \in \mathsf{D}(R) \mid \mathrm{supp}(X) \subseteq V\}$ is a localizing $\otimes$-ideal of T.