## DUALITIES IN TRIANGULATED CATEGORIES
## ASSIGNMENT 2. DUE: 18TH APRIL 2024

(1) Let $\mathsf{T}$ be a tensor-triangulated category, and suppose that $X \in \mathsf{T}$ is rigid. Prove that the natural map
$$F(Y, \mathbb{1}) \otimes X \to F(Y, X)$$
is an isomorphism for all $Y \in \mathsf{T}$.

(2) This exercise concerns the construction of Brown-Comenetz duals in tensor-triangulated categories. These are certain 'designer' objects which play an important role in stable homotopy theory. Let $\mathsf{T}$ be a rigidly-compactly generated tensor-triangulated category.
   (1) Let $C \in \mathsf{T}^{\mathrm{c}}$. Show that there exists an object $\mathbb{I}_C \in \mathsf{T}$ such that
   $$\mathrm{Hom}_{\mathsf{T}}(-, \mathbb{I}_C) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathsf{T}}(C, -), \mathbb{Q}/\mathbb{Z}).$$
   (2) Define a functor $I_C \colon \mathsf{T}^{\mathrm{op}} \to \mathsf{T}$ by $I_C(-) := F(-, \mathbb{I}_C)$. Prove that $I_C(X) \simeq F(F(C, X), \mathbb{I}_{\mathbb{1}})$.
   (3) Let $X \in \mathsf{T}$. Prove that if $I_{\mathbb{1}}(X) \simeq 0$, then $X \simeq 0$. (*Hint:* Recall that $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for abelian groups, so that if $M \in \mathsf{Mod}(\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \simeq 0$, then $M \simeq 0$.)
   (4) Let $\mathcal{X}$ be a set of a compact objects and suppose that if $X \in \mathcal{X}$ and $C \in \mathsf{T}^{\mathrm{c}}$, then $C \otimes X \in \mathcal{X}$. Consider the set
   $$\mathcal{X}^{\perp_{\mathbb{Z}}} := \{Y \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(\Sigma^i X, Y) \simeq 0 \text{ for all } X \in \mathcal{X} \text{ and } i \in \mathbb{Z}\}.$$
   Show that if $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp_{\mathbb{Z}}}$, then $X \otimes Y \simeq 0$. Deduce that if $Y \in \mathcal{X}^{\perp_{\mathbb{Z}}}$, then $I_C(Y) \in \mathcal{X}^{\perp_{\mathbb{Z}}}$ for all $C \in \mathsf{T}^{\mathrm{c}}$.
   (5) Consider $\mathsf{T} = \mathsf{D}(R)$ for a commutative ring $R$. What is $\mathbb{I}_R$?