DUALITIES IN TRIANGULATED CATEGORIES
ASSIGNMENT 3. DUE: 22ND MAY 2024

(1) Let R be a commutative ring and $I = (x_1, \cdots, x_n)$ be a finitely generated ideal of R. Prove that if $(x_1, \cdots, x_n)$ is a regular sequence, then R/I is a compact object of $\mathsf{D}(R)$. (Recall that a sequence $(x_1, \cdots, x_n)$ is regular if x_1 is not a zero-divisor of R, and x_i is not a zero-divisor of $R/(x_1, \cdots, x_{i-1})$ for all $i \geq 2$.)

(2) Let T be a tensor-triangulated category which is compactly generated by rigid objects. Suppose that L is a monoidal localization of T, and write $\mathcal{S}$ for the full subcategory of L-local objects.
   (a) Prove that the natural map $\alpha_X \colon L\mathbb{1} \otimes X \to LX$ is an equivalence for all rigid X.
   (b) Prove that the following conditions are equivalent:
      (i) The natural map $\alpha_X \colon L\mathbb{1} \otimes X \to LX$ is an equivalence for all $X \in \mathsf{T}$.
      (ii) $i \colon \mathcal{S} \to \mathsf{T}$ preserves coproducts.
      (iii) $L \colon \mathsf{T} \to \mathsf{T}$ preserves coproducts.
      (iv) $\mathcal{S}$ is a localizing subcategory of T.
   (c) Give an example of a smashing localization.
   (d) Prove that $L \colon \mathsf{T} \to \mathcal{S}$ preserve compacts if L is smashing. Does this still hold if L is not smashing?